

Advanced Computer Graphics

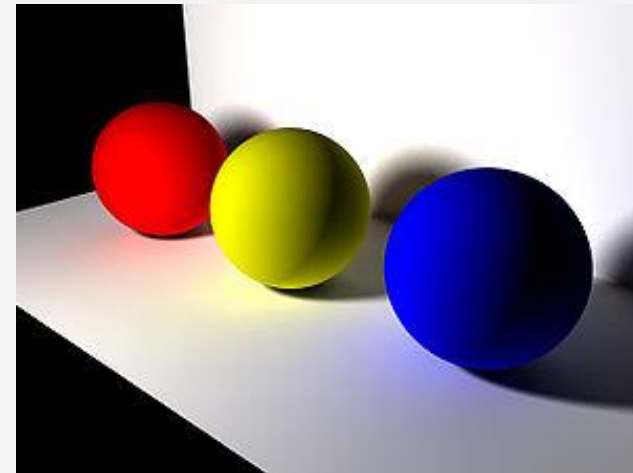
Radiosity 1

Matthias Teschner



Motivation

- Rendering algorithm
- Global illumination approach
 - Global solution of a linear system
 - Considers global illumination (direct and indirect)
- View-independent solution
- Limited to Lambertian surfaces
 - Diffuse global Illumination

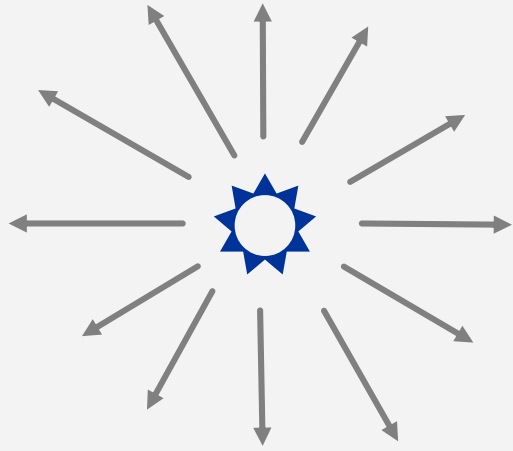


Wikipedia: Radiosity
(Computergraphik)

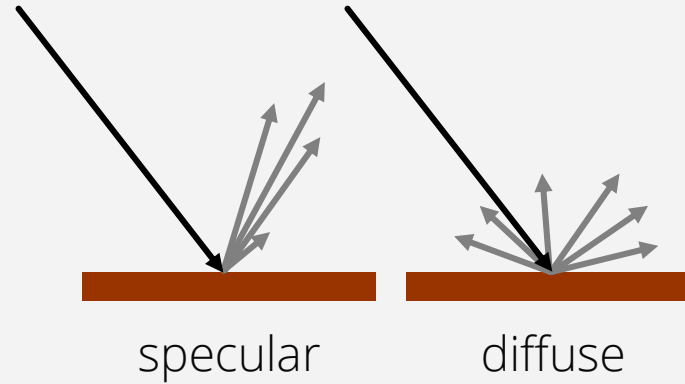
Outline

- Context
- Governing Equation
- System
- Solver
- Discussion

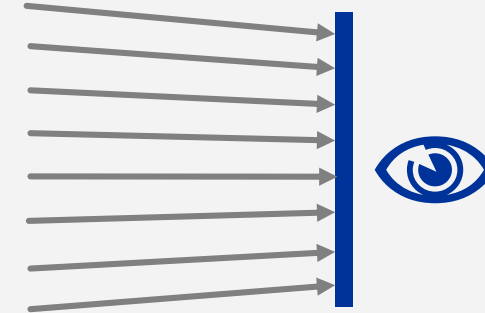
Towards Image Generation



Light is emitted at light sources



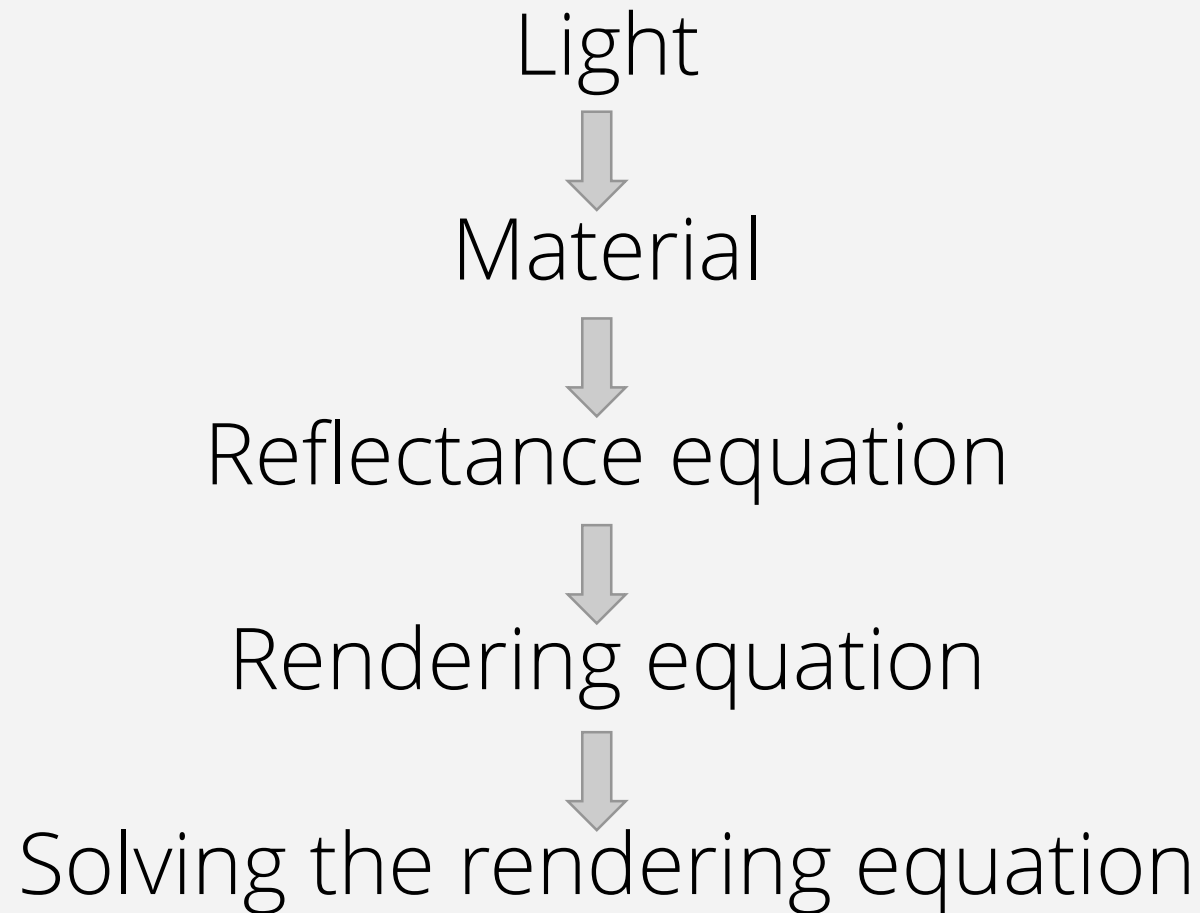
Light is absorbed and scattered at surfaces



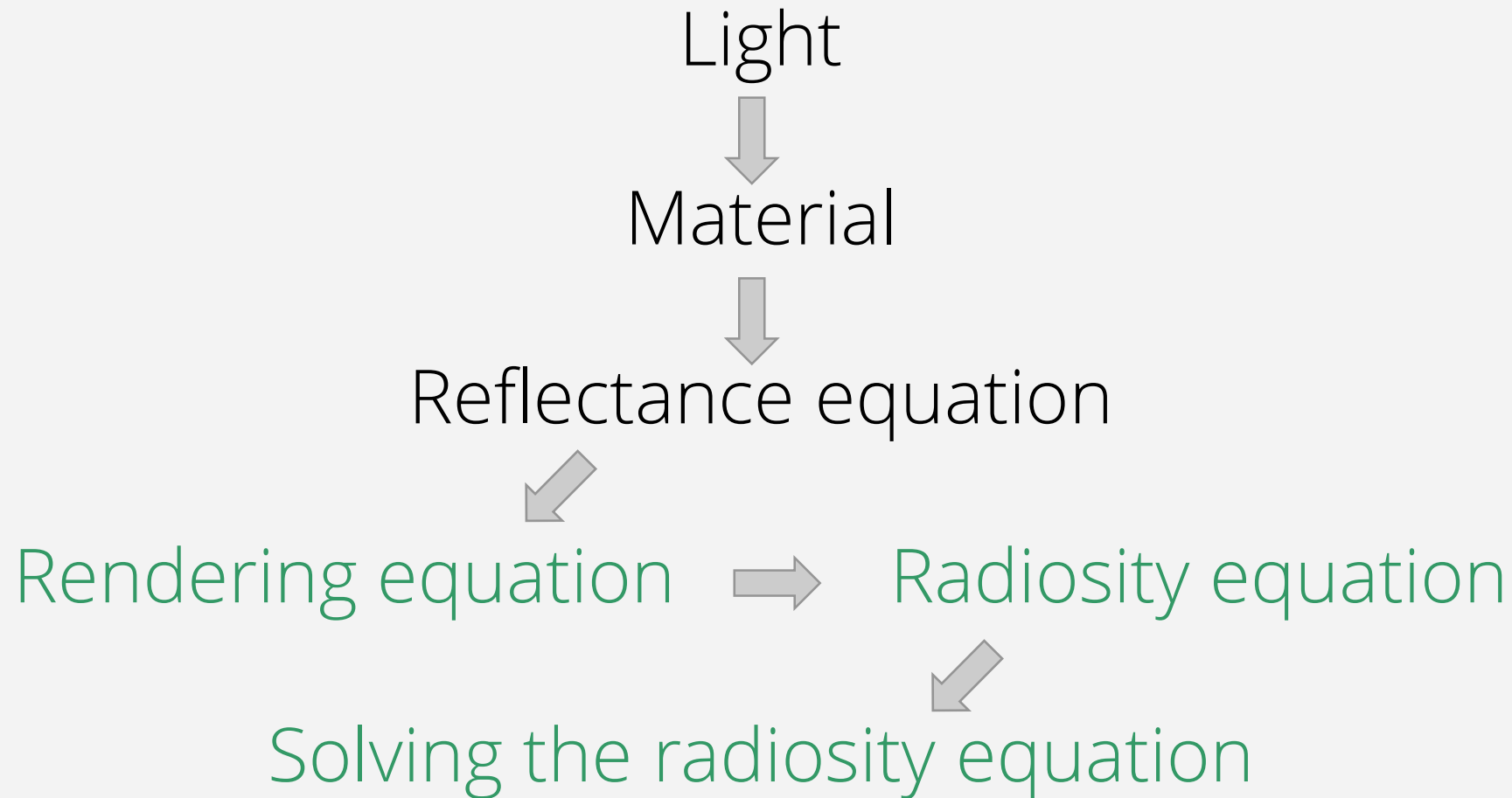
Cameras capture light

- Sources emit light
- Surfaces absorb and reflect light
- Rendering algorithms compute light at sensors

Towards Image Generation



Radiosity – Rendering Algorithm



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Notation

- Incident radiance onto a point

$$L_i(\mathbf{p}, \omega_i), L(\mathbf{p} \leftarrow \omega_i)$$

- Exitant radiance from a point

$$L_o(\mathbf{p}, \omega_o), L(\mathbf{p} \rightarrow \omega_o)$$

- BRDF at a point

$$f_r(\mathbf{p}, \omega_i, \omega_o), f_r(\mathbf{p}, \omega_i \leftrightarrow \omega_o)$$

Reflectance Equation

- Relation between irradiance and exitant radiance

$$dL(\mathbf{p} \rightarrow \omega_o) = f_r(\mathbf{p}, \omega_i \leftrightarrow \omega_o) dE(\mathbf{p} \leftarrow \omega_i)$$

- Irradiance is induced by radiance

$$dL(\mathbf{p} \rightarrow \omega_o) = f_r(\mathbf{p}, \omega_i \leftrightarrow \omega_o) L(\mathbf{p} \leftarrow \omega_i) \cos(\omega_i, \mathbf{n}_p) d\omega_i$$

- Integration over the hemisphere \Rightarrow reflectance equation

$$L(\mathbf{p} \rightarrow \omega_o) = \int_{\Omega} f_r(\mathbf{p}, \omega_i \leftrightarrow \omega_o) L(\mathbf{p} \leftarrow \omega_i) \cos(\omega_i, \mathbf{n}_p) d\omega_i$$

- Reflectance equation establishes a relation between incident and exitant radiance

Reflectance vs. Rendering Equation

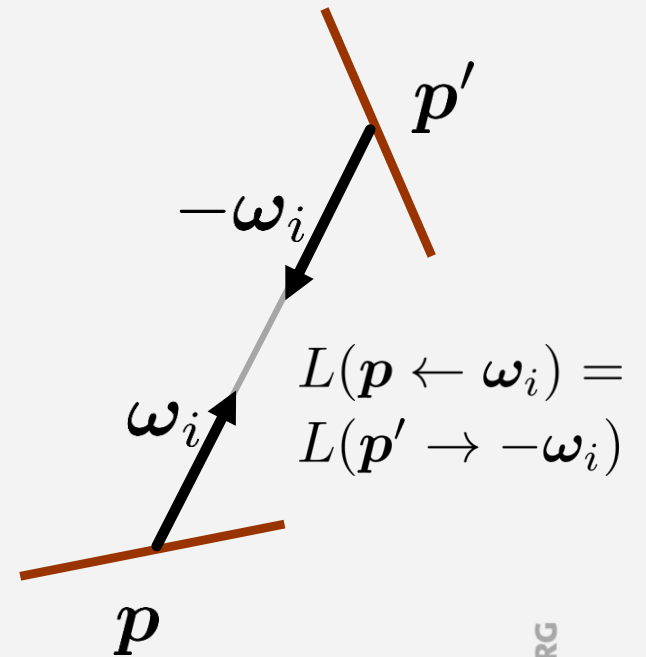
- Reflectance equation relates incident and exitant radiance at surfaces
- Rendering equation incorporates emissive surfaces, i.e. light sources

$$L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) + \int_{\Omega} f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \mathbf{n}_p) d\omega_i$$

- Exitant radiance is the sum of emitted and reflected radiance

Ray-Casting Operator

- Incident radiance $L(\mathbf{p} \leftarrow \boldsymbol{\omega}_i)$ at a point \mathbf{p} is equal to the exitant radiance $L(\mathbf{p}' \rightarrow -\boldsymbol{\omega}_i)$ from another point \mathbf{p}'
- Ray-casting operator $\mathbf{p}' = \mathbf{r}_c(\mathbf{p}, \boldsymbol{\omega}_i)$
 - Nearest intersection from \mathbf{p} into direction $\boldsymbol{\omega}_i$
- Radiance
 - $L(\mathbf{p} \leftarrow \boldsymbol{\omega}_i) = L(\mathbf{r}_c(\mathbf{p}, \boldsymbol{\omega}_i) \rightarrow -\boldsymbol{\omega}_i)$
 - $L(\mathbf{p} \leftarrow \boldsymbol{\omega}_i) = L(\mathbf{p}' \rightarrow -\boldsymbol{\omega}_i)$
 - If $\mathbf{r}_c(\mathbf{p}, \boldsymbol{\omega}_i)$ does not exist, $L(\mathbf{p} \leftarrow \boldsymbol{\omega}_i)$ is user-defined, e.g. emission from sky



Rendering Equation

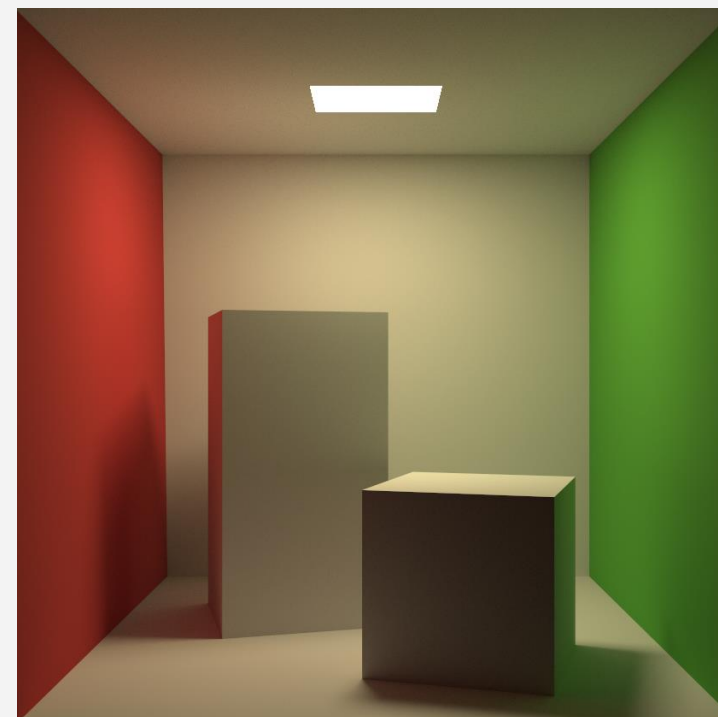
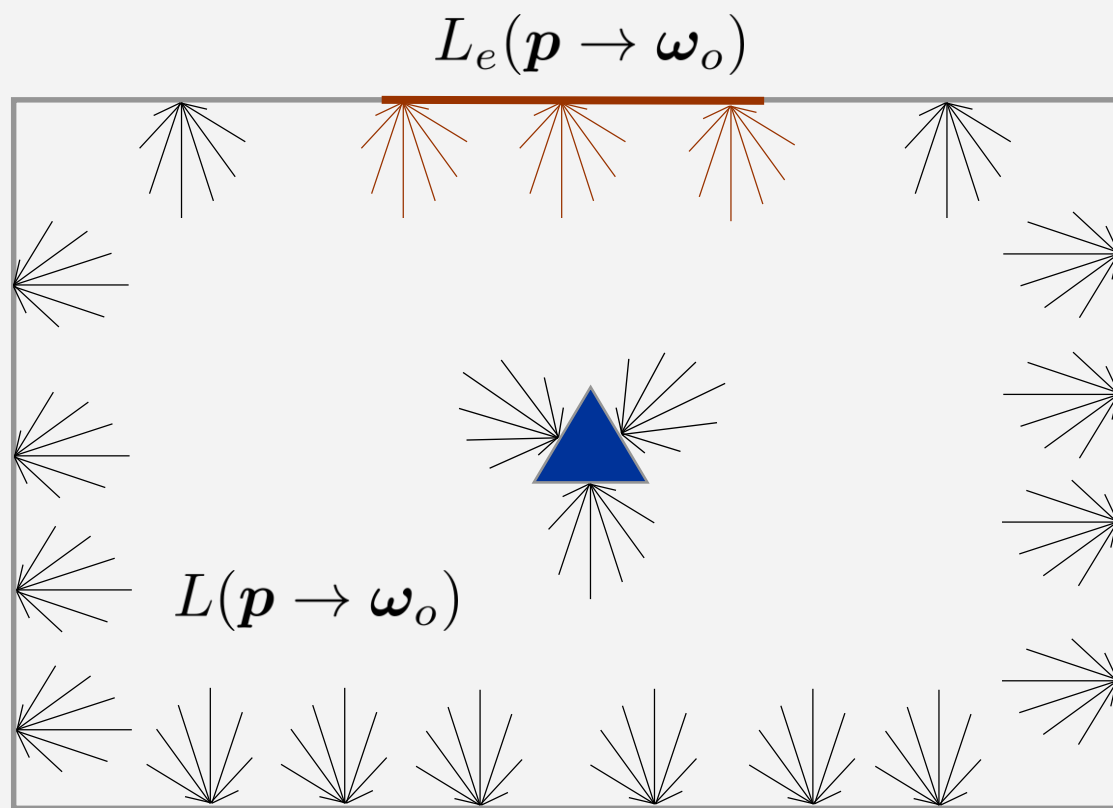
- $L(\mathbf{p} \rightarrow \omega_o) = L_e(\mathbf{p} \rightarrow \omega_o) + \int_{\Omega} f_r(\mathbf{p}, \omega_i \leftrightarrow \omega_o) L(\mathbf{p}' \rightarrow -\omega_i) \cos(\omega_i, \mathbf{n}_p) d\omega_i$
- Establishes relations among exitant radiances
- Governs the computation of exitant radiances from all scene points into all directions



[Akenine-Möller et al.]

Solution of the Rendering Equation

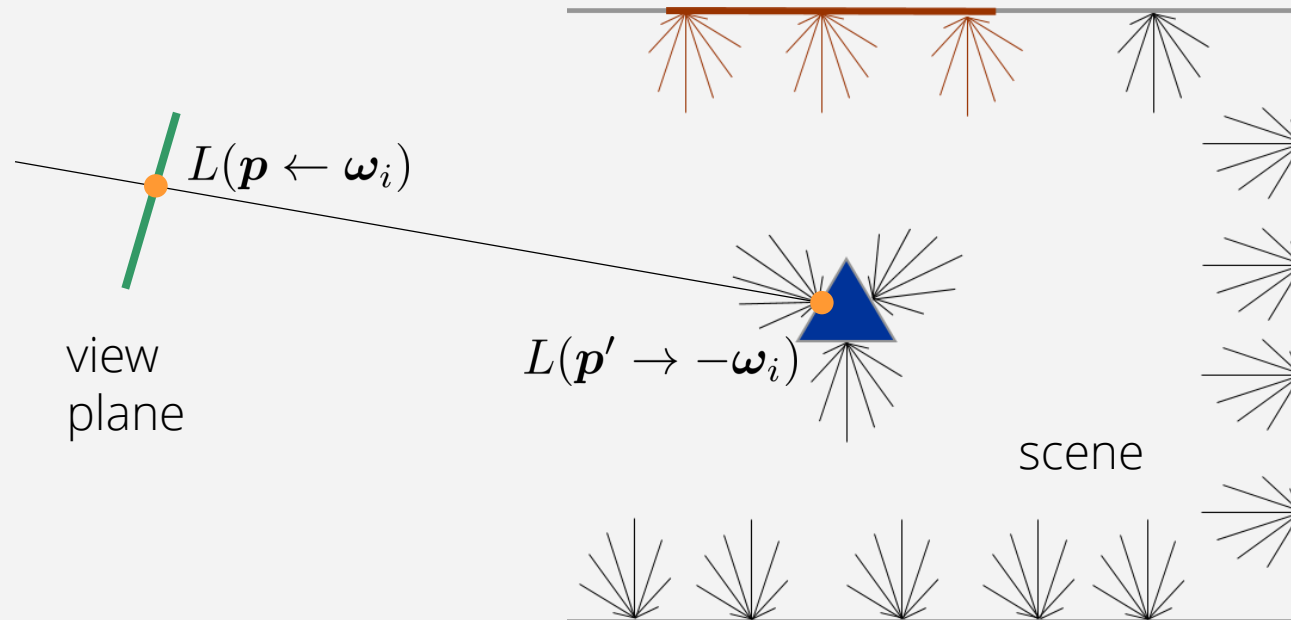
- Exitant radiances from all scene points into all directions



Cornell box

Rendering of the Solution

- At an arbitrarily placed and oriented sensor
 - Cast a ray through position \mathbf{p} in an image plane into direction $\boldsymbol{\omega}_i$
 - Lookup $L(\mathbf{p} \leftarrow \boldsymbol{\omega}_i) = L(\mathbf{r}_c(\mathbf{p}, \boldsymbol{\omega}_i) \rightarrow -\boldsymbol{\omega}_i) = L(\mathbf{p}' \rightarrow -\boldsymbol{\omega}_i)$



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Simplified Setting

- Lambertian material
 - Exitant radiance independent from direction
 - Radiance into arbitrary direction can be computed from radiosity $L(\mathbf{p} \rightarrow \omega_o) = \frac{B(\mathbf{p})}{\pi}$
 - Discretized scene representation with faces, e.g., triangles
 - Assume constant radiosity per face
- ⇒ Problem is simplified to n radiosity values for n faces
- ⇒ n instances of the rendering equation govern the solution

Goal - System of Governing Equations

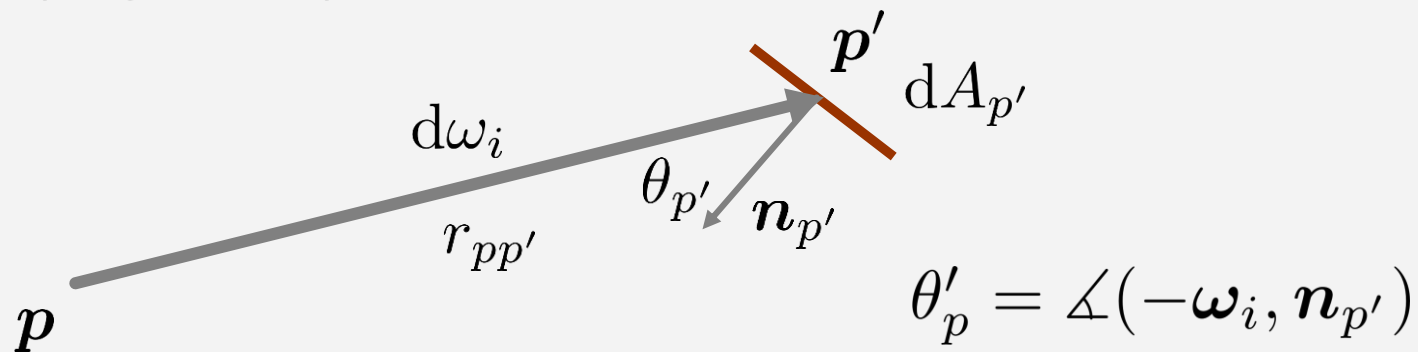
- Simplified setting results in a linear system with unknown radiosity values B_i at faces

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn-1} & a_{nn} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

- Radiosity B_i at face i depends on radiosities B_j at faces j which are visible from face i

Hemispherical and Area Form

- Differential solid angle corresponds to a differential surface area
- If an infinitesimally small area $dA_{p'}$ at position \mathbf{p}' converges to zero, then the solid angle $d\omega_i$ also converges to zero and the relation $d\omega_i = \frac{\cos(-\boldsymbol{\omega}_i, \mathbf{n}_{p'})}{r_{pp'}^2} dA_{p'}$ is correct in the limit



Hemispherical and Area Form

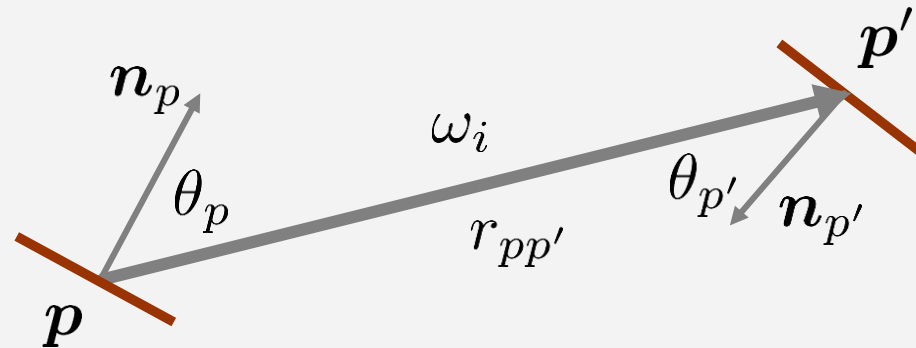
- Hemispherical form of the rendering equation

$$L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) + \int_{\Omega} f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{p}' \rightarrow -\boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \mathbf{n}_p) d\omega_i$$

can be written in area form

$$L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) +$$

$$\int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{p}' \rightarrow -\boldsymbol{\omega}_i) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_{p'})}{r_{pp'}^2} dA_{p'}$$



$$\theta_p = \angle(\boldsymbol{\omega}_i, \mathbf{n}_p)$$

$$\theta_{p'} = \angle(-\boldsymbol{\omega}_i, \mathbf{n}_{p'})$$

Area Form of the Rendering Equation

- Integral over all differential surface areas obtained from the ray-casting operator $\mathbf{p}' = \mathbf{r}_c(\mathbf{p}, \boldsymbol{\omega}_i)$

$$L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) +$$

$$\int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{p}' \rightarrow -\boldsymbol{\omega}_i) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_{p'})}{r_{pp'}^2} dA_{p'}$$

- Integral over all differential surface areas \mathbf{x} of a scene

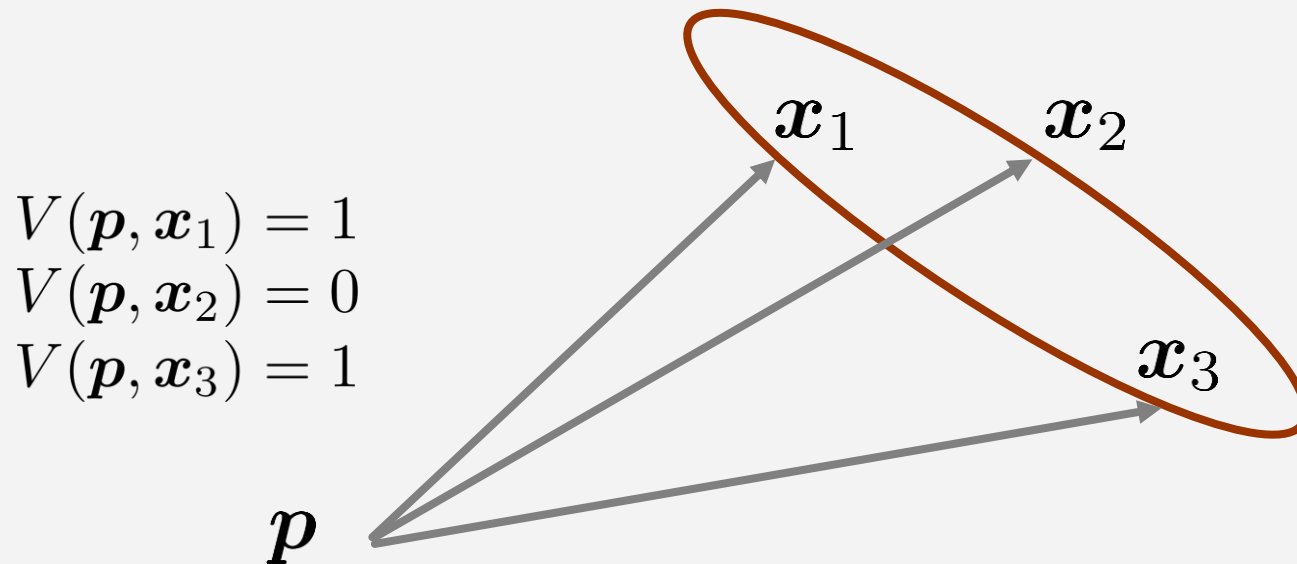
$$L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) +$$

$$\int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{x} \rightarrow -\boldsymbol{\omega}_i) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$$

visibility
function

Visibility Function

- $\int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{x} \rightarrow -\boldsymbol{\omega}_i) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$
- Position \mathbf{x} contributes to the integral, if it is visible from \mathbf{p}



Radiosity Integral Equation

- Rendering equation

$$L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) +$$

$$\int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\mathbf{x} \rightarrow -\boldsymbol{\omega}_i) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$$

- Radiance can be computed from radiosity

for Lambertian surfaces: $L(\mathbf{p} \rightarrow \boldsymbol{\omega}_o) = \frac{B(\mathbf{p})}{\pi}$

- Radiosity equation

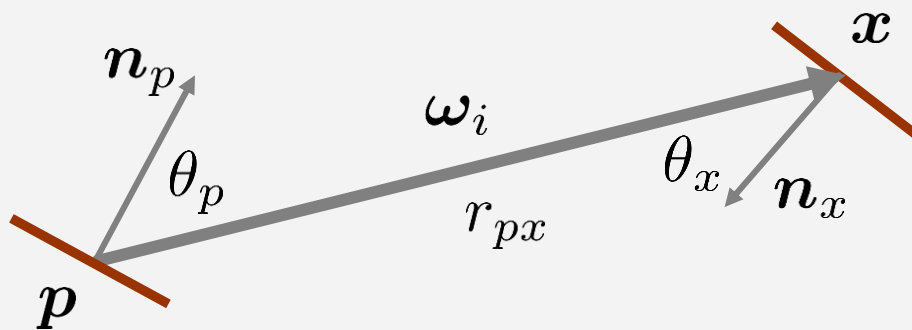
$$B(\mathbf{p}) = B_e(\mathbf{p}) + \int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) B(\mathbf{x}) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$$

$$B(\mathbf{p}) = B_e(\mathbf{p}) + \frac{\rho(\mathbf{p})}{\pi} \int_S B(\mathbf{x}) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$$

Constant BRDF
for Lambertian
surfaces

Kernel

- $K(\mathbf{p}, \mathbf{x}) = V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{\pi r_{px}^2} \geq 0$
- Radiosity equation $B(\mathbf{p}) = B_e(\mathbf{p}) + \rho(\mathbf{p}) \int_S K(\mathbf{p}, \mathbf{x}) B(\mathbf{x}) dA_x$
- Kernel weights the contribution of patch \mathbf{x} for the radiosity at patch \mathbf{p} and vice versa

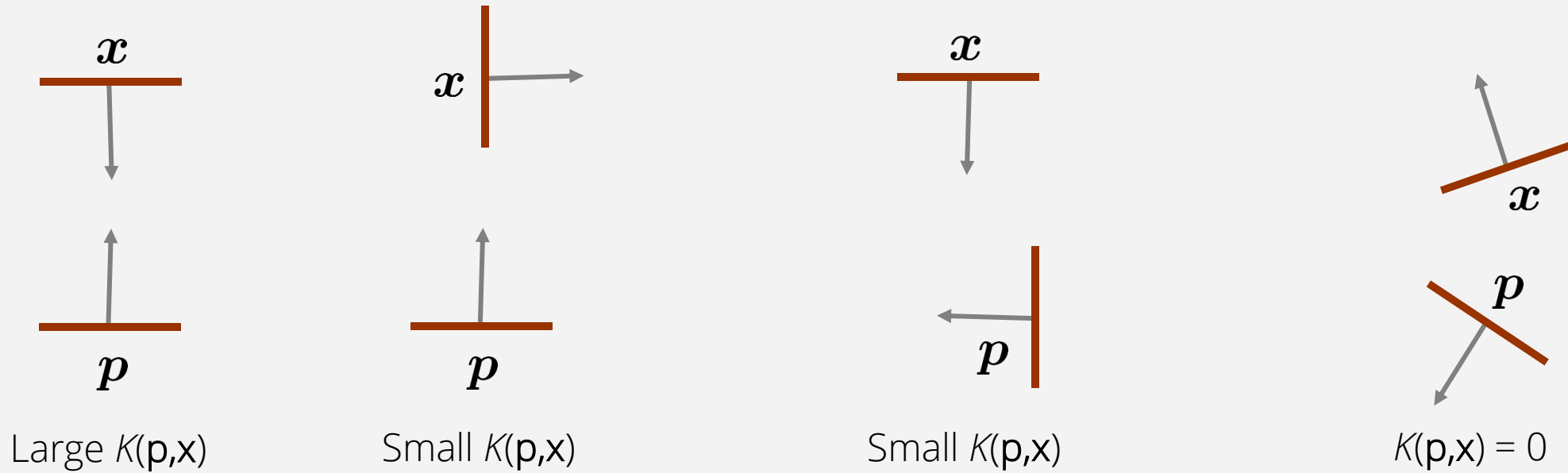


$$\theta_p = \angle(\boldsymbol{\omega}_i, \mathbf{n}_p)$$

$$\theta_x = \angle(-\boldsymbol{\omega}_i, \mathbf{n}_x)$$

$K(\mathbf{p}, \mathbf{x})$ gets larger
- if \mathbf{p} and \mathbf{x} are oriented towards each other
- if \mathbf{p} and \mathbf{x} are closer to each other

Kernel



– Indicates the “importance” of patch x for patch p

Discretization

- Continuous form of the radiosity equation

- $B(\mathbf{p}) = B_e(\mathbf{p}) + \rho(\mathbf{p}) \int_S K(\mathbf{p}, \mathbf{x}) B(\mathbf{x}) dA_x$ Radiosity at points

- Infinite number of equations for infinite number of unknowns

- Discretization (Finite Element Method)

- One unknown per face / triangle / finite element

- n equations for n unknowns

$$\begin{pmatrix} 1 - \rho_1 K_{11} A_1 & -\rho_1 K_{12} A_2 & \dots & -\rho_1 K_{1n} A_n \\ -\rho_2 K_{21} A_1 & 1 - \rho_2 K_{22} A_2 & \dots & -\rho_2 K_{2n} A_n \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_n K_{n1} A_1 & \dots & \dots & 1 - \rho_n K_{nn} A_n \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} B_{e1} \\ B_{e2} \\ \vdots \\ B_{en} \end{pmatrix} \quad \text{Radiosity at faces}$$

Advanced Computer Graphics

Radiosity 2

Matthias Teschner



Discretization

- Continuous form of the radiosity equation
 - $B(\mathbf{p}) = B_e(\mathbf{p}) + \rho(\mathbf{p}) \int_S K(\mathbf{p}, \mathbf{x}) B(\mathbf{x}) dA_x$ Radiosity at points
 - Infinite number of equations for infinite number of unknowns
- Discretization (Finite Element Method)
 - One unknown per face / triangle / finite element
 - n equations for n unknowns

$$\begin{pmatrix} 1 - \rho_1 K_{11} A_1 & -\rho_1 K_{12} A_2 & \dots & -\rho_1 K_{1n} A_n \\ -\rho_2 K_{21} A_1 & 1 - \rho_2 K_{22} A_2 & \dots & -\rho_2 K_{2n} A_n \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_n K_{n1} A_1 & \dots & \dots & 1 - \rho_n K_{nn} A_n \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} B_{e1} \\ B_{e2} \\ \vdots \\ B_{en} \end{pmatrix} \quad \text{Radiosity at faces}$$

From Differential Areas to Finite Areas

- Start with, e.g., a triangulated scene representation
- Assume constant radiosity over area A_i : $B(\mathbf{x}_i) = \text{const} = B_i$
- Assume constant reflectance over area A_i : $\rho(\mathbf{x}_i) = \text{const} = \rho_i$
- Integrate radiosity over a face i with area A_i
- Radiosity equation for face i

$$\int_{S_i} B(\mathbf{x}_i) dA_{x_i} = \int_{S_i} B_e(\mathbf{x}_i) dA_{x_i} + \int_{S_i} \rho(\mathbf{x}_i) \int_S K(\mathbf{x}_i, \mathbf{x}) B(\mathbf{x}) dA_x dA_{x_i}$$

$$A_i B_i = A_i B_{ei} + \rho_i \int_{S_i} \int_S K(\mathbf{x}_i, \mathbf{x}) B(\mathbf{x}) dA_x dA_{x_i}$$

- B_i, B_{ei} are radiosity and emitted radiosity per face i

From Differential Areas to Finite Areas

- $\int_S K(\mathbf{x}_i, \mathbf{x})B(\mathbf{x})dA_x$ is an integral over all faces of a scene
- Can be written as $\sum_j \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j)B(\mathbf{x}_j)dA_{x_j}$
 - Integral over a face j , summed over all faces
- Radiosity equation

$$A_i B_i = A_i B_{ei} + \rho_i \int_{S_i} \int_S K(\mathbf{x}_i, \mathbf{x}) B(\mathbf{x}) dA_x dA_{x_i}$$

$$B_i = B_{ei} + \rho_i \frac{1}{A_i} \int_{S_i} \sum_j \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) B(\mathbf{x}_j) dA_{x_j} dA_{x_i} \quad \text{Division by } A_i$$

$$B_i = B_{ei} + \rho_i \frac{1}{A_i} \sum_j \int_{S_i} \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) B(\mathbf{x}_j) dA_{x_j} dA_{x_i}$$

From Differential Areas to Finite Areas

$$B_i = B_{ei} + \rho_i \frac{1}{A_i} \sum_j \int_{S_i} \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) B(\mathbf{x}_j) dA_{x_j} dA_{x_i}$$

- Constant radiosity over area A_j , i.e. $B(\mathbf{x}_j) = \text{const} = B_j$

$$B_i = B_{ei} + \rho_i \frac{1}{A_i} \sum_j B_j \int_{S_i} \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) dA_{x_j} dA_{x_i}$$

- Form factor: $F_{ij} = \frac{1}{A_i} \int_{S_i} \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) dA_{x_j} dA_{x_i}$
- Almost discretized radiosity equation

$$B_i = B_{ei} + \sum_j \rho_i F_{ij} B_j$$

Form Factor – A First Approximation

$$F_{ij} = \frac{1}{A_i} \int_{S_i} \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) dA_{x_j} dA_{x_i}$$

- Assume constant kernel for two faces i and j :

$$K(\mathbf{x}_i, \mathbf{x}_j) = \text{const} = K_{ij} \quad \text{Bad for pairs of faces that see each other only partially}$$

$$F_{ij} = \frac{1}{A_i} K_{ij} \int_{S_i} \int_{S_j} dA_{x_j} dA_{x_i} = K_{ij} A_j$$

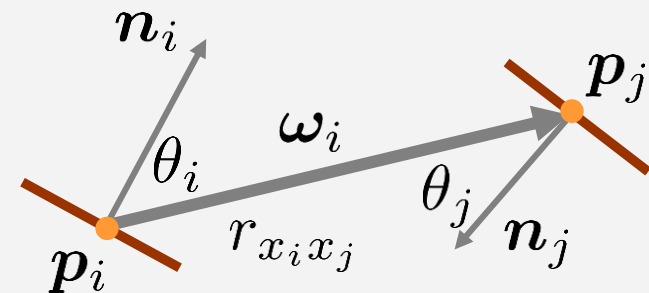
- Choose representative positions $\mathbf{p}_i, \mathbf{p}_j$ on faces i and j

$$F_{ij} = V(\mathbf{p}_i, \mathbf{p}_j) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_i) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_j)}{\pi r_{\mathbf{p}_i \mathbf{p}_j}^2} A_j$$

Non-zero for faces
that “see” each other

$$\theta_i = \angle(\boldsymbol{\omega}_i, \mathbf{n}_i)$$

$$\theta_j = \angle(-\boldsymbol{\omega}_i, \mathbf{n}_j)$$



Discretization of the Radiosity Equation

- Continuous form, per surface position

$$B(\mathbf{p}) = B_e(\mathbf{p}) + \frac{\rho(\mathbf{p})}{\pi} \int_S B(\mathbf{x}) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$$

- Discretized form, per face / triangle Finite Element Method

$$B_i = B_{ei} + \sum_j \rho_i F_{ij} B_j$$

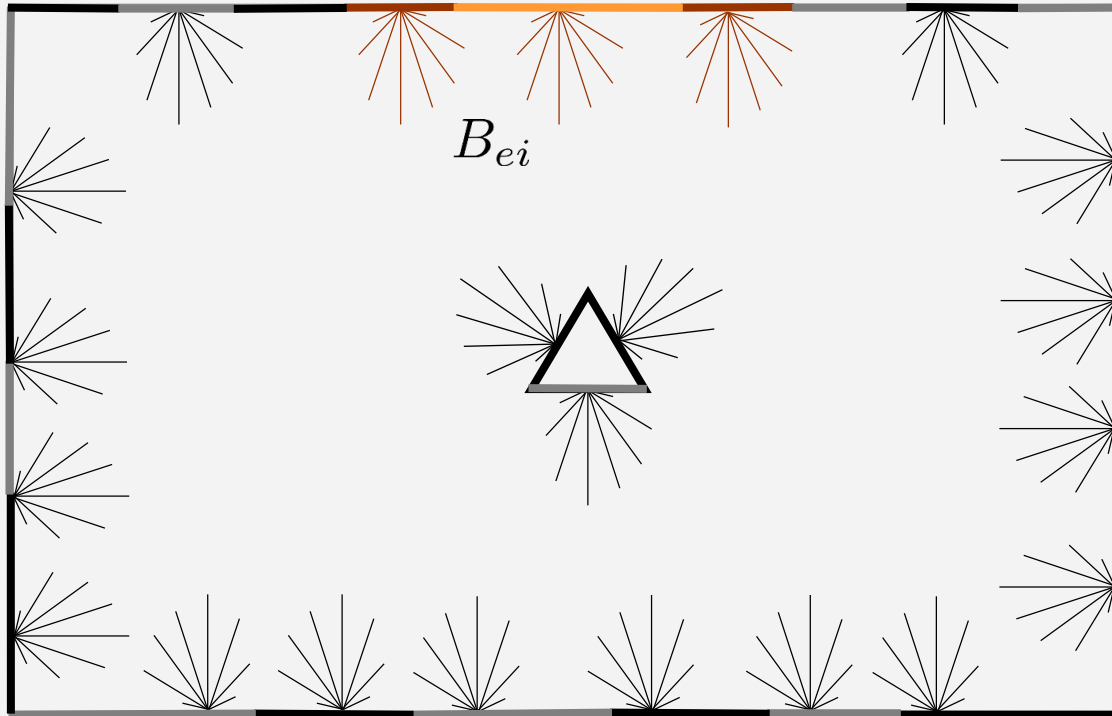
$$B_i - \sum_j \rho_i F_{ij} B_j = B_{ei}$$

- B_{ei} is a source, i.e. the **known** emitted radiosity at face i
- B_i, B_j are **unknown** radiosities at faces i and j
- ρ_i, F_{ij} are **known** coefficients

System of Linear Equations

- n equations for n unknown radiosities at n faces

$$B_i - \sum_j \rho_i F_{ij} B_j = B_{ei}$$



$$B_i - \sum_j \rho_i F_{ij} B_j = 0$$

If $B_{ei} = 0$ for all faces, the solution would be $B_i = 0$ for all faces.

System of Linear Equations

$$\begin{pmatrix} 1 - \rho_1 F_{11} & -\rho_1 F_{12} & \dots & -\rho_1 F_{1n} \\ -\rho_2 F_{21} & 1 - \rho_2 F_{22} & \dots & -\rho_2 F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_n F_{n1} & \dots & -\rho_n F_{nn-1} & 1 - \rho_n F_{nn} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} B_{e1} \\ B_{e2} \\ \vdots \\ B_{en} \end{pmatrix}$$

Matrix with known coefficients, reflectances and form factors. Indirect illumination. Describes, how faces illuminate each other.

Unknown radiosities.

Known source terms. Direct illumination.

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Solving the Linear System

- Typically with iterative schemes, e.g. relaxed Jacobi
 - Initialize, e.g., $B_i^0 = 0$ Superscript numbers indicate the solver iteration
 - Iteratively update $B_i^{l+1} = B_i^l + \frac{\lambda_i}{1-\rho_i F_{ii}} (B_{ei} - (B_i^l - \sum_j \rho_i F_{ij} B_j^l))$
- Intuition
 - Changes from B_i^l to B_i^{l+1} are proportional to $B_{ei} - (B_i^l - \sum_j \rho_i F_{ij} B_j^l)$
 - If $B_{ei} - (B_i^l - \sum_j \rho_i F_{ij} B_j^l) = 0$, i.e. $B_i^l - \sum_j \rho_i F_{ij} B_j^l = B_{ei}$, the solver has converged and $B_i^{l+1} = B_i^l$

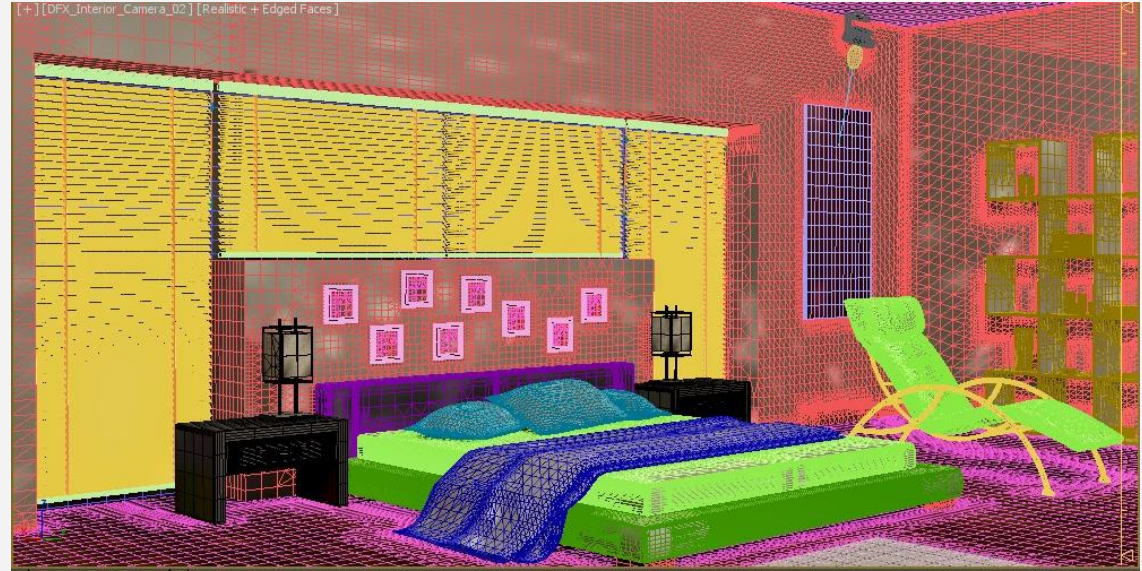
Summary

- Scene modeling / meshing
- Computation of form factors for pairs of patches
- Solve linear system
- Set up a camera
- Project scene onto view plane / cast rays into the scene
- Lookup radiosity / reconstruct radiance per pixel

Meshing Example



Low / adaptive resolution



High resolution

Rendering of the Solution



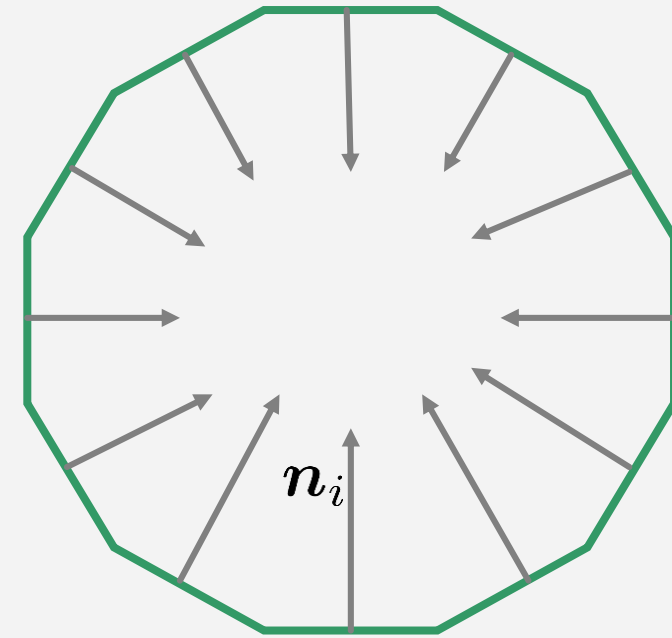
Final rendering from an arbitrary position and orientation.

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Form Factor Computation

- Important and expensive
- Worst case
 - Inside of a convex polygon
 - All faces see each other
 - Complexity of a naive form factor computation is quadratic in the number of faces
 - System matrix is fully filled with non-zero entries

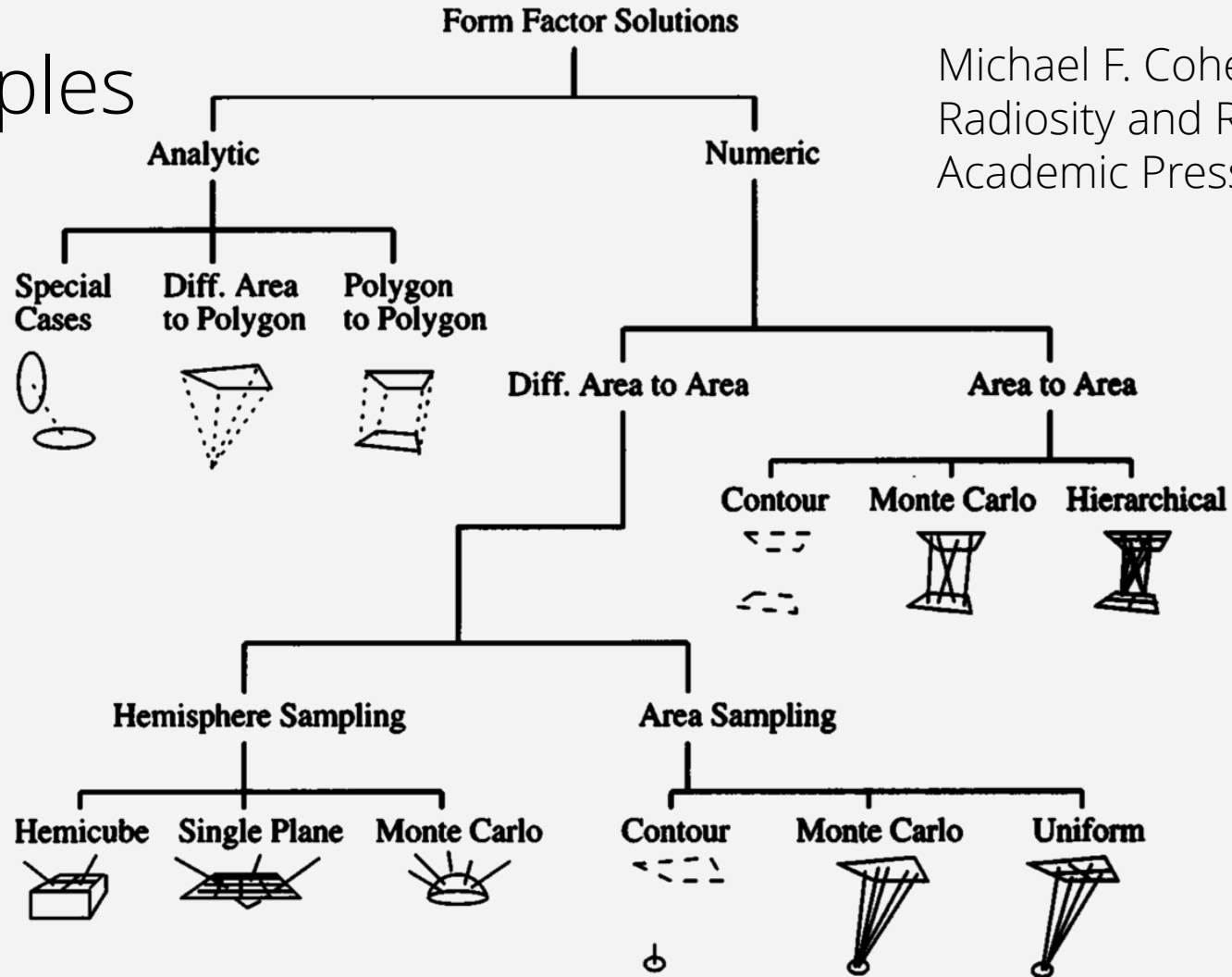


$$\forall i, \forall j : F_{ij} \neq 0$$

Form Factor Solutions

– Examples

Michael F. Cohen, John R. Wallace:
Radiosity and Realistic Image Synthesis.
Academic Press Professional, Boston.



Form Factor Properties

– Positive

$$F_{ij} = \frac{1}{A_i} \int_{S_i} \int_{S_j} V(\mathbf{x}_i, \mathbf{x}_j) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_i) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_j)}{\pi r_{\mathbf{x}_i \mathbf{x}_j}^2} dA_{x_j} dA_{x_i} \geq 0$$

– Reciprocity relation

$$K(\mathbf{x}_i, \mathbf{x}_j) = V(\mathbf{x}_i, \mathbf{x}_j) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_i) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_j)}{\pi r_{\mathbf{x}_i \mathbf{x}_j}^2} = K(\mathbf{x}_j, \mathbf{x}_i)$$

$$\begin{aligned} A_i F_{ij} &= A_i \frac{1}{A_i} \int_{S_i} \int_{S_j} K(\mathbf{x}_i, \mathbf{x}_j) dA_{x_j} dA_{x_i} \\ &= A_j \frac{1}{A_j} \int_{S_j} \int_{S_i} K(\mathbf{x}_j, \mathbf{x}_i) dA_{x_i} dA_{x_j} \\ &= A_j F_{ji} \end{aligned}$$

Form Factor Properties

$$\begin{aligned}\sum_j F_{ij} &= \frac{1}{A_i} \int_{S_i} \sum_j \int_{S_j} V(\mathbf{x}_i, \mathbf{x}_j) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_i) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_j)}{\pi r_{\mathbf{x}_i \mathbf{x}_j}^2} dA_{x_j} dA_{x_i} \\ &= \frac{1}{A_i} \int_{S_i} \int_{2\pi} \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_i)}{\pi} d\omega_i dA_{x_i} \\ &= \frac{1}{A_i} \int_{S_i} \frac{\pi}{\pi} dA_{x_i} = 1\end{aligned}$$

$$\sum_j F_{ij} = 1 \Rightarrow \rho_i \sum_j F_{ij} \leq 1 \Rightarrow \rho_i \sum_{j \neq i} F_{ij} \leq 1 - \rho_i F_{ii}$$

- Important for the convergence of iterative solvers
 - Diagonally dominant system matrix
 - Sum of magnitudes of non-diagonal entries per row is smaller than the magnitude of the diagonal entry
 - Surface properties influence convergence

System Notation

$$\begin{pmatrix} 1 - \rho_1 F_{11} & -\rho_1 F_{12} & \dots & -\rho_1 F_{1n} \\ -\rho_2 F_{21} & 1 - \rho_2 F_{22} & \dots & -\rho_2 F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_n F_{n1} & \dots & -\rho_n F_{nn-1} & 1 - \rho_n F_{nn} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix} = \begin{pmatrix} B_{e1} \\ B_{e2} \\ \vdots \\ B_{en} \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} \rho_1 F_{11} & \rho_1 F_{12} & \dots & \rho_1 F_{1n} \\ \rho_2 F_{21} & \rho_2 F_{22} & \dots & \rho_2 F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_n F_{n1} & \dots & \rho_n F_{nn-1} & \rho_n F_{nn} \end{pmatrix}$$

$$(\mathbf{I} - \mathbf{F})\mathbf{B} = \mathbf{B}_e \quad \mathbf{B} = \mathbf{B}_e + \mathbf{F}\mathbf{B}$$

System Interpretation

– Radiosity equation per point

$$- B(\mathbf{p}) = B_e(\mathbf{p}) + \int_S f_r(\mathbf{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) B(\mathbf{x}) V(\mathbf{p}, \mathbf{x}) \frac{\cos(\boldsymbol{\omega}_i, \mathbf{n}_p) \cos(-\boldsymbol{\omega}_i, \mathbf{n}_x)}{r_{px}^2} dA_x$$

– System of per-face discretized radiosity equations

$$- \mathbf{B} = \mathbf{B}_e + \mathbf{F}\mathbf{B}$$

– \mathbf{B} overall radiosity at all faces

– \mathbf{B}_e radiosity at all faces due to emission

– $\mathbf{F}\mathbf{B}$ radiosity at all faces due to the reflection of incident flux from all faces

System Solution

$$(\mathbf{I} - \mathbf{F})\mathbf{B} = \mathbf{B}_e$$

$$\mathbf{B} = (\mathbf{I} - \mathbf{F})^{-1}\mathbf{B}_e$$

– Neumann series

$$(\mathbf{I} - \mathbf{F})^{-1} = \sum_{k=0}^{\infty} \mathbf{F}^k$$

The inverse does not always exist. In particular, there is no solution for unphysical settings.

$$\mathbf{B} = \mathbf{B}_e + \mathbf{F}\mathbf{B}_e + \mathbf{F}\mathbf{F}\mathbf{B}_e + \mathbf{F}\mathbf{F}\mathbf{F}\mathbf{B}_e + \dots$$

Terms in the Neumann Series

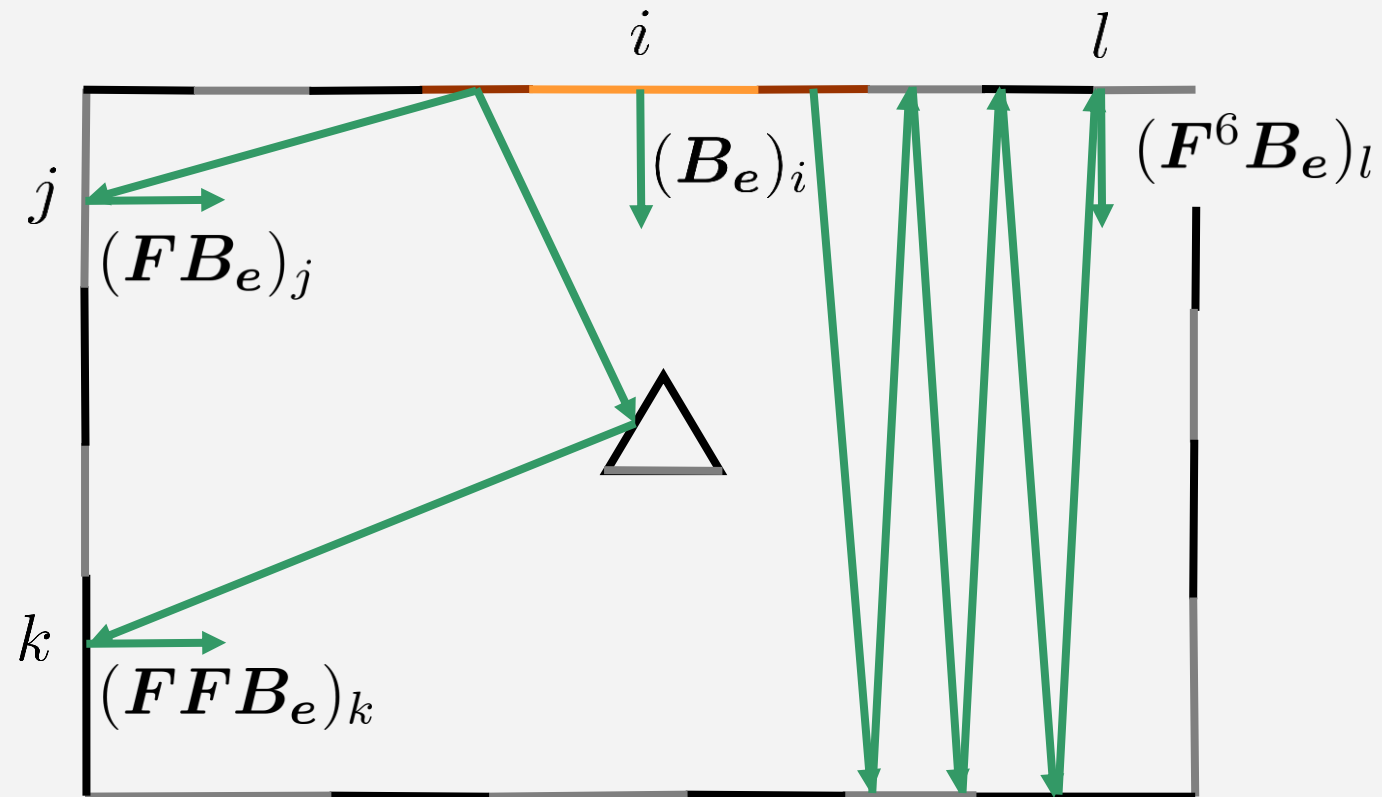
- $B = B_e + FB_e + FFB_e + FFFB_e + \dots$
 - B_e emitted radiosity
 - FB_e reflected radiosity due to emitted radiosity B_e
(emitted radiosity after one bounce at a surface)
 - FFB_e reflected radiosity due to radiosity that
was reflected due to emitted radiosity B_e
(emitted radiosity after two bounces at surfaces)
 - $FFFB_e \dots$

Terms in the Neumann Series

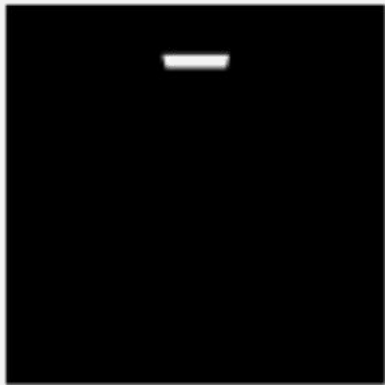
- $B = B_e + FB_e + FFB_e + FFFB_e + \dots$
 - B_e contribution of emitted light to the solution
 - FB_e contribution of emitted light after one bounce
 - FFB_e contribution of emitted light after two bounces
 - $FFFB_e$ contribution of emitted light after three bounces

Terms in the Neumann Series

- Example contributions to terms



Visualizing the Neumann Series



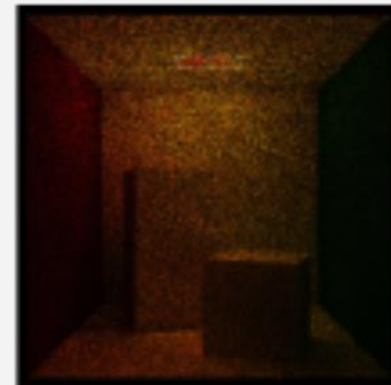
B_e



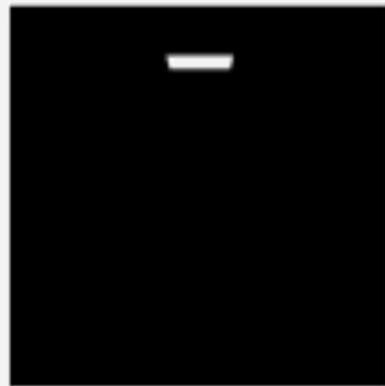
FB_e



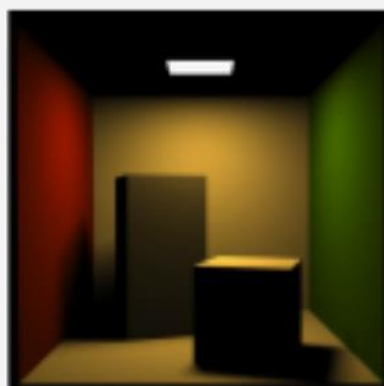
FFB_e



$FFFB_e$



B_e



$B_e + FB_e$



$B_e + FB_e + FFB_e$



\dots

Terms in the Neumann Series

$$\mathbf{B} = \mathbf{B}_e + \mathbf{F}\mathbf{B}_e + \mathbf{F}\mathbf{F}\mathbf{B}_e + \mathbf{F}\mathbf{F}\mathbf{F}\mathbf{B}_e + \dots$$

- Emissive patches are important
- Patches that have large form factors with respect to emissive patches are important
- Pairs of patches with large form factors F_{ij} are potentially important
- Highly reflective patches with large reflectance coefficients ρ_i are potentially important

Jacobi Solver

- Jacobi with, e.g., $\frac{\lambda_i}{1-\rho_i F_{ii}} = 1$: $B_i^{l+1} = B_{ei} + \sum_j \rho_i F_{ij} B_j^l$
 - $\mathbf{B}^{l+1} = \mathbf{B}_e + \mathbf{F}\mathbf{B}^l$
- Iterations
 - $\mathbf{B}^0 = \mathbf{0}$
 - $\mathbf{B}^1 = \mathbf{B}_e$
 - $\mathbf{B}^2 = \mathbf{B}_e + \mathbf{F}\mathbf{B}^1 = \mathbf{B}_e + \mathbf{F}\mathbf{B}_e$
 - $\mathbf{B}^3 = \mathbf{B}_e + \mathbf{F}\mathbf{B}^2 = \mathbf{B}_e + \mathbf{F}(\mathbf{B}_e + \mathbf{F}\mathbf{B}_e)$
 $= \mathbf{B}_e + \mathbf{F}\mathbf{B}_e + \mathbf{F}\mathbf{F}\mathbf{B}_e$
- Intuition does not necessarily apply to other solvers

Solver Convergence

- Radiosity contributions B_e FB_e FFB_e $FFFFB_e$ should get smaller with each iteration
 - Some faces in a scene should partially absorb flux
 - Faces should not generate flux, i.e. $\rho_i > 1$

$$\mathbf{F} = \begin{pmatrix} \rho_1 F_{11} & \rho_1 F_{12} & \dots & \rho_1 F_{1n} \\ \rho_2 F_{21} & \rho_2 F_{22} & \dots & \rho_2 F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_n F_{n1} & \dots & \rho_n F_{nn-1} & \rho_n F_{nn} \end{pmatrix} \quad \begin{array}{l} 0 \leq \rho_i \leq 1 \\ \sum_j F_{ij} = 1 \end{array}$$