Advanced Computer Graphics
Aliasing

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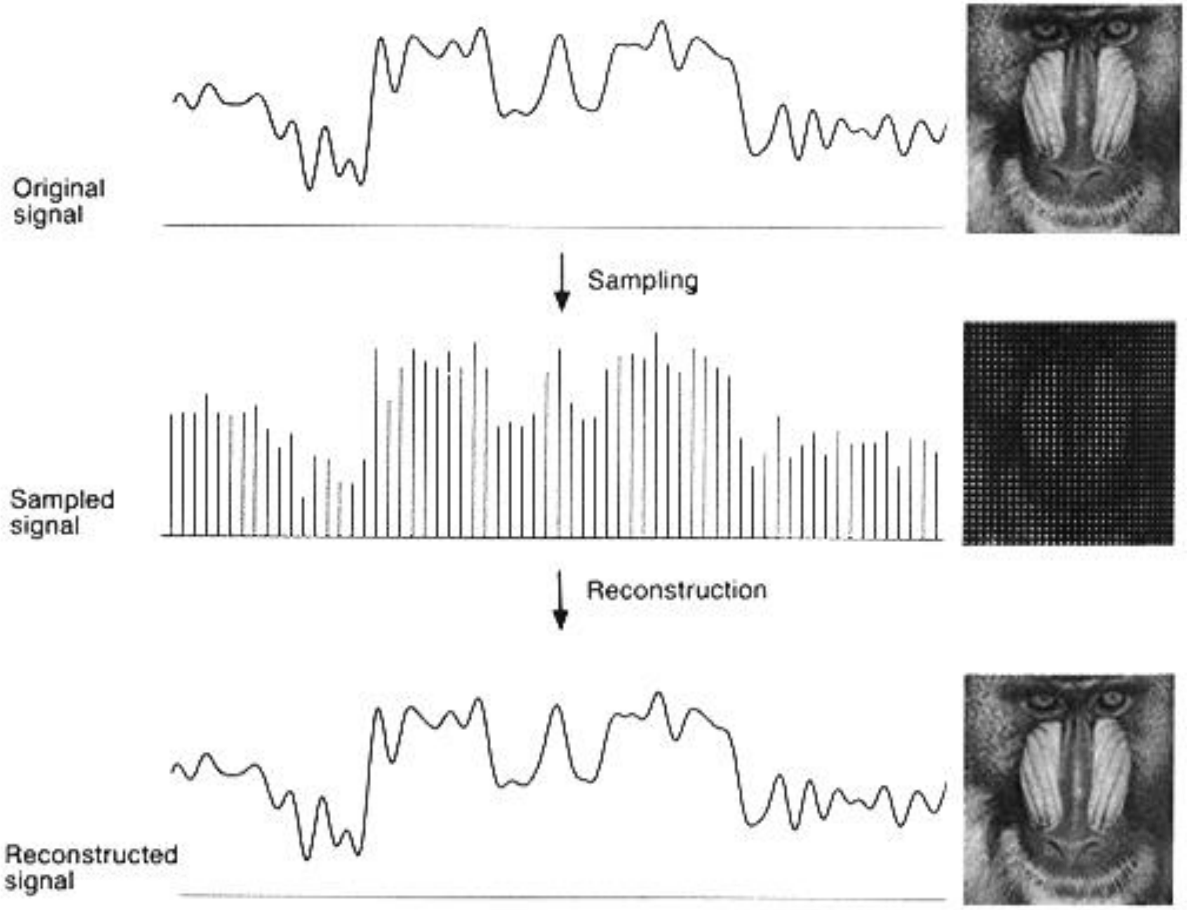
UNI FREIBURG

Outline

- Motivation
- Fourier analysis
- Filtering
- Sampling
- Reconstruction / aliasing
- Antialiasing

Motivation

- Sampling and reconstruction
- Inappropriate sampling can cause artifacts in reconstructed functions



[Foley, van Dam, Feiner, Huges]

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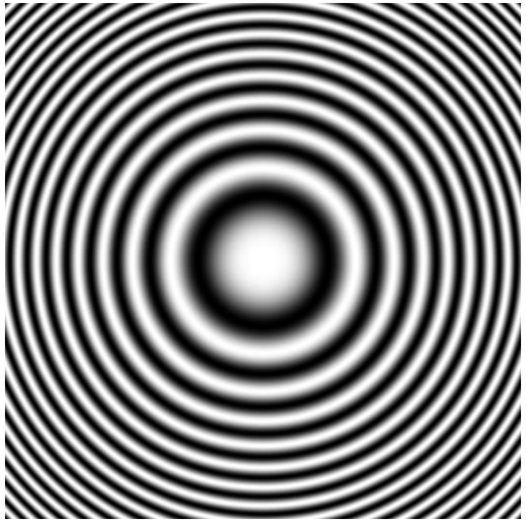
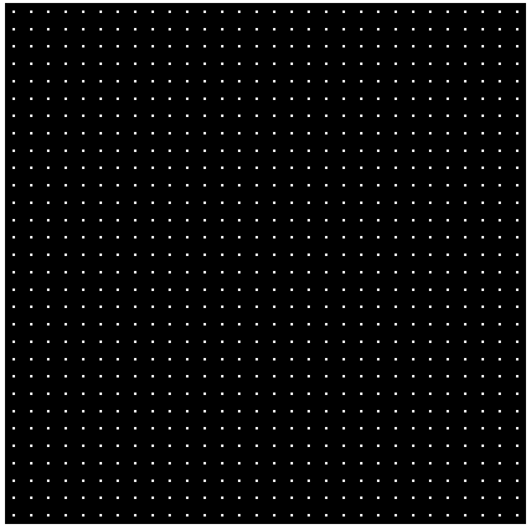
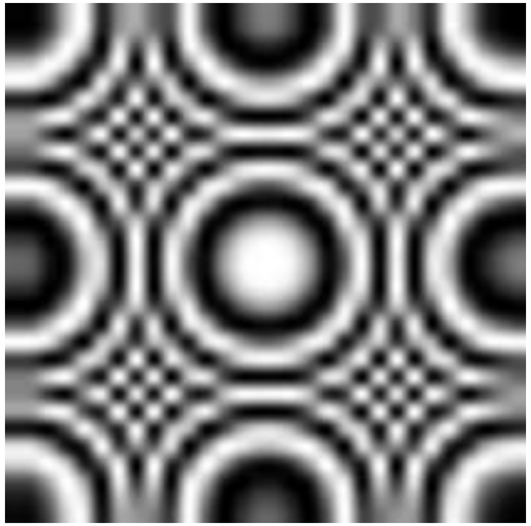
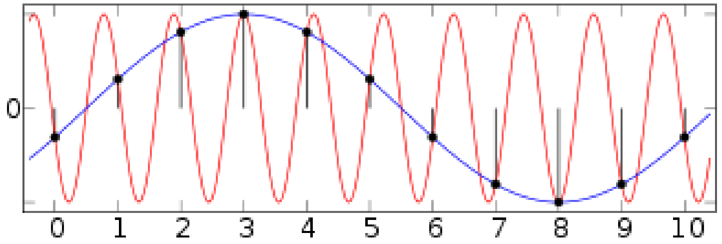
Motivation

– Aliasing artifacts, e. g. Moiré pattern


Original signal


Sampled signal


Reconstructed signal

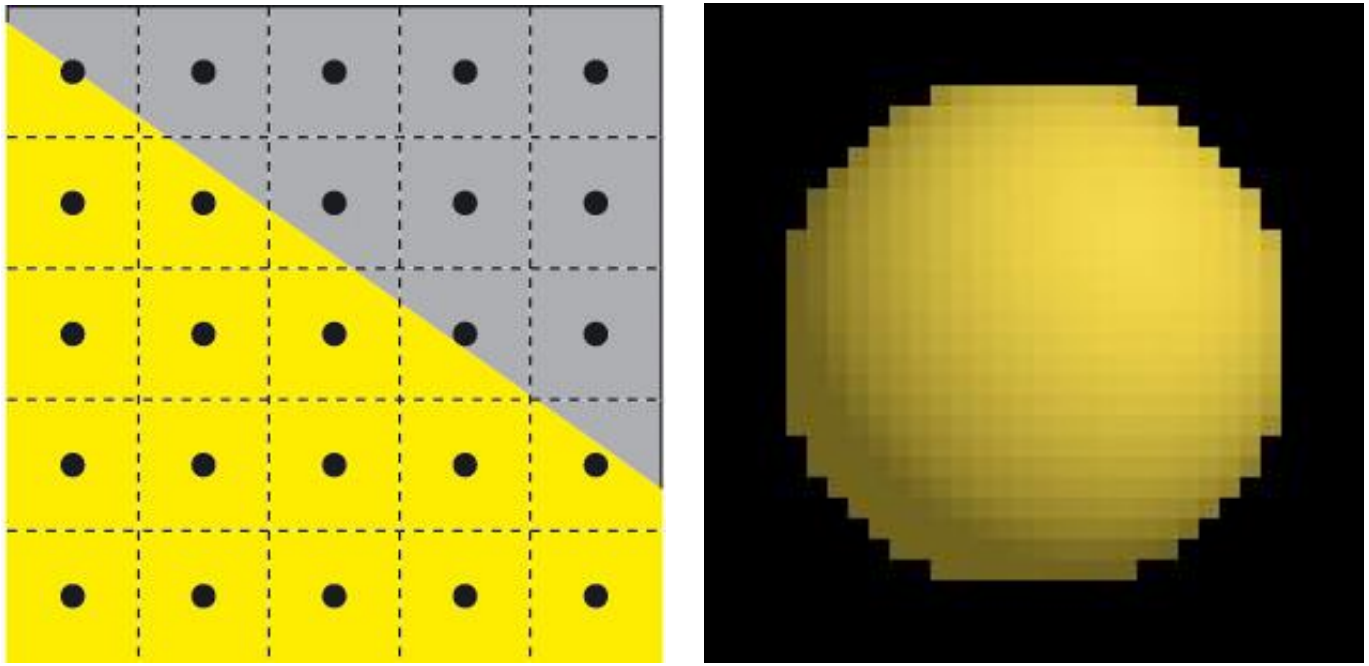


Red - original signal
Black dots - samples
Blue - reconstructed signal

[Wikipedia: Alias-Effekt]

Motivation

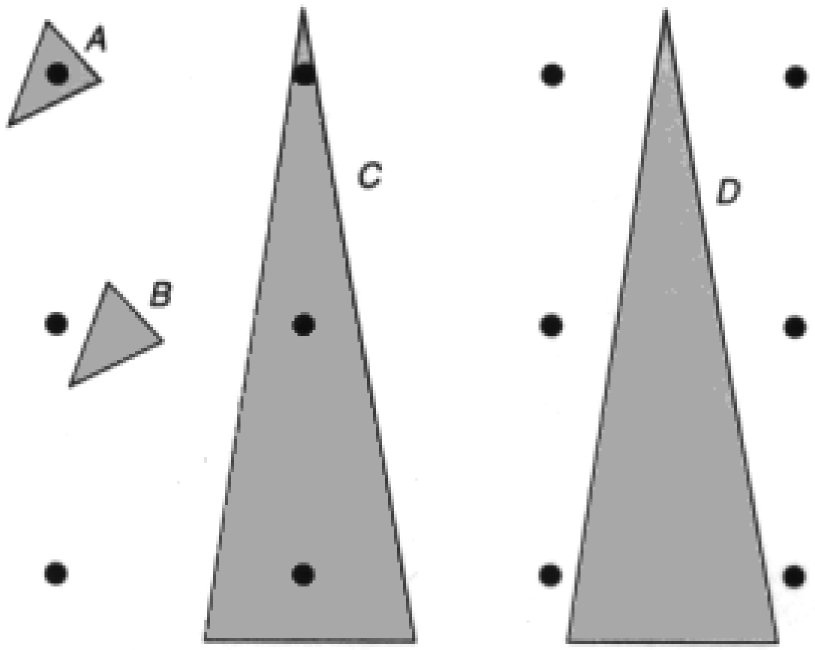
– Aliasing artifacts, e. g. jaggies



[Suffern]

Motivation

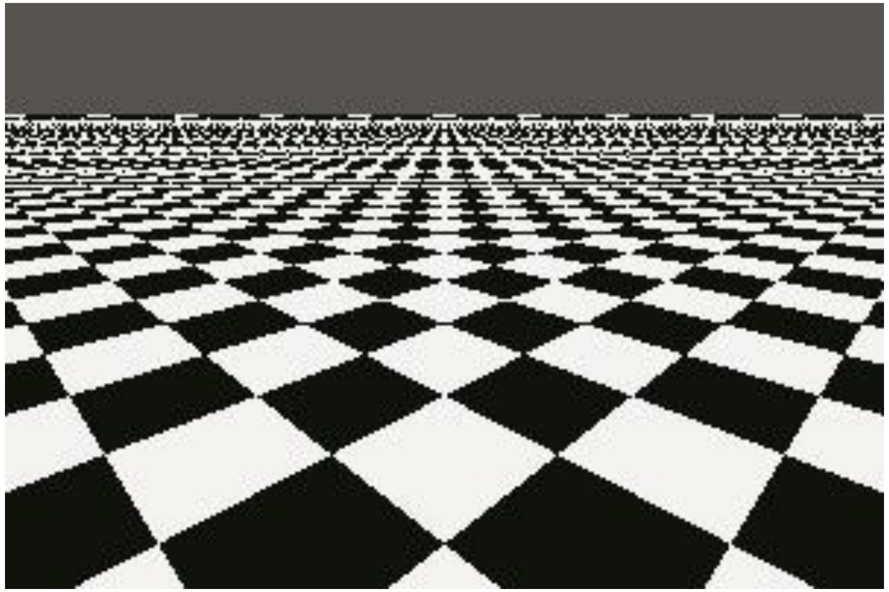
– Aliasing artifacts, e. g. missing details



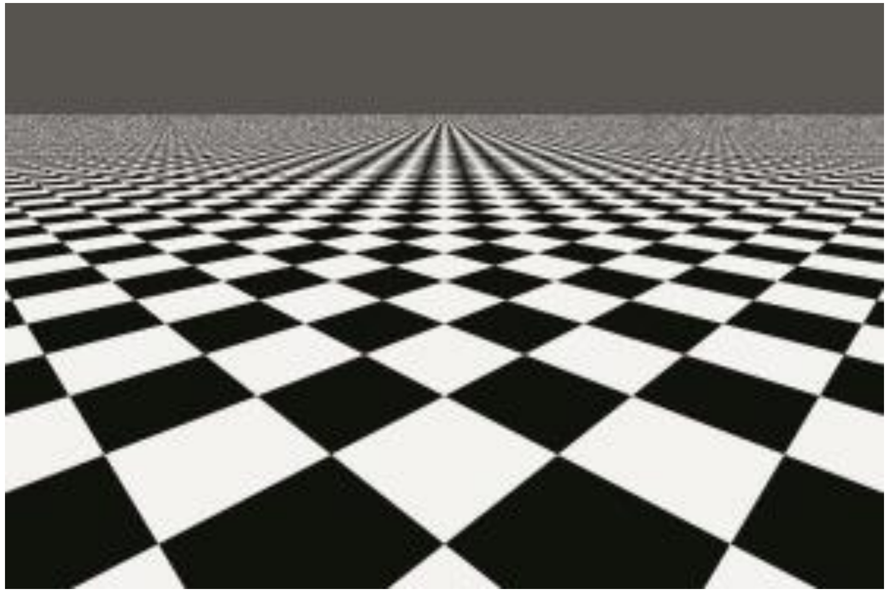
[Foley, van Dam, Feiner, Huges]

Motivation

- Antialiasing
  - Reduction of erroneous patterns



Aliasing artifacts



[Suffern]

Regular artifacts replaced by less disturbing noise

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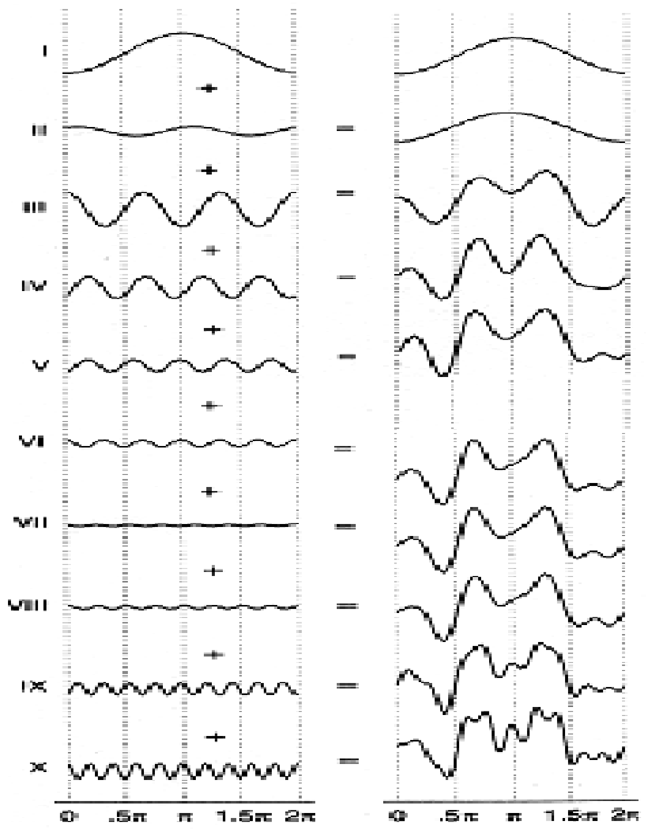
Spectrum of a Function

- Fourier transform
  - Decomposes a function into weighted sum of shifted sinusoids
  - Computes amplitude and phase shift of frequencies contained in the function
  - Transforms from the spatial domain to the frequency domain

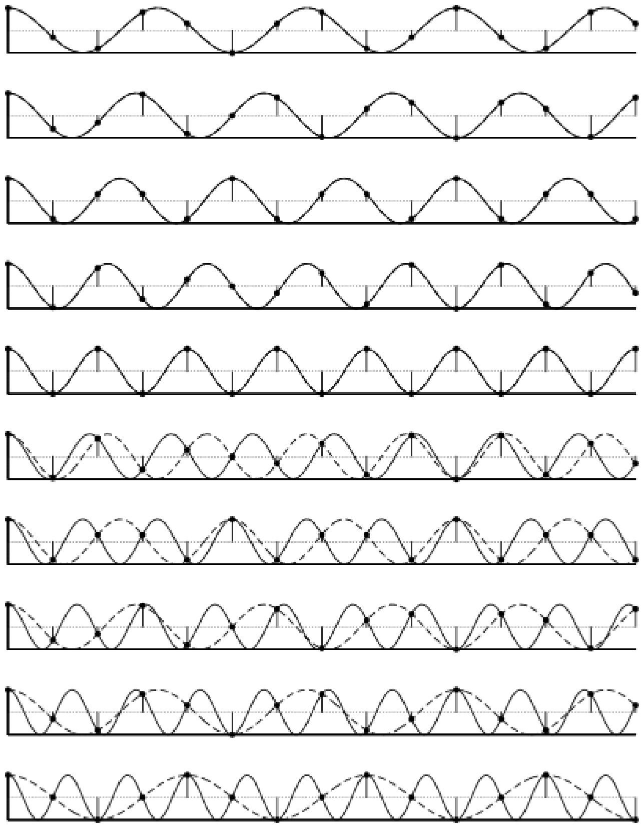
$$\mathfrak{F}\{f(x)\} = F(\omega) \qquad \mathfrak{F}^{-1}\{F(\omega)\} = f(x)$$



[Foley, van Dam, Feiner, Huges]

Spectrum of a Function - Motivation

– Analysis in the frequency domain allows to understand aliasing
– Aliasing is then reduced by
  – Adapting the sampling
  – Filtering the original signal (for textures)



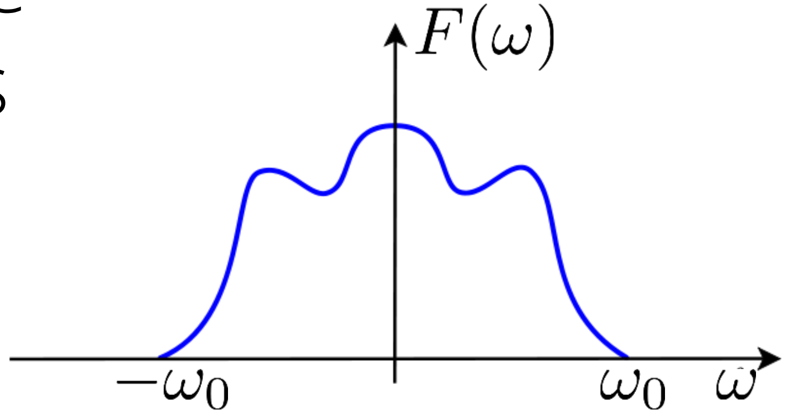
Appropriate sampling frequencies

Inappropriate sampling frequencies

[Wikipedia: Nyquist-Shannon-Abtasttheorem]

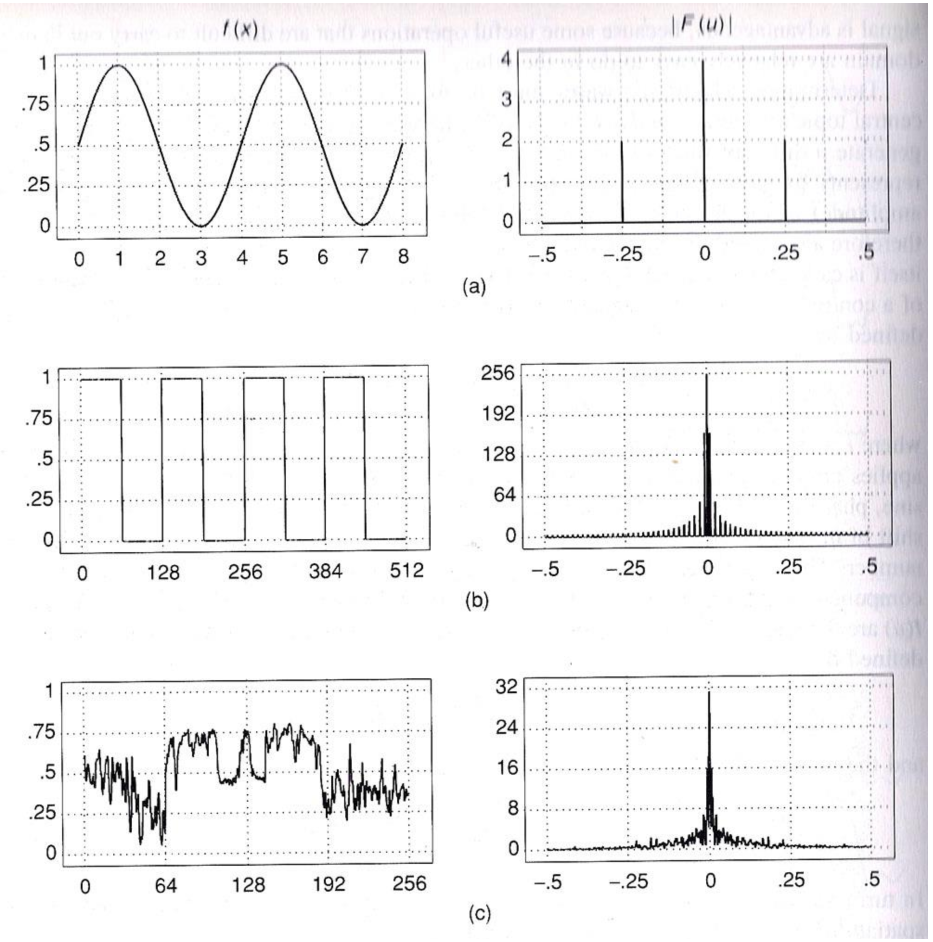
Spectrum of a Function - Motivation

- Sampling and reconstruction can be analyzed in the frequency domain
- A band-limited function with $F(\omega) = 0$ for all $\omega > \omega_0$ has to be sampled with a frequency $\omega_{\text{sampling}} > 2\omega_0$ in order to be able to reconstruct the original function from the samples (Nyquist-Shannon sampling theorem)
- Nyquist frequency ω_0
- Nyquist rate ω_{sampling}



Fourier Transform - Examples

– Signals in spatial and frequency domain



[Foley, van Dam, Feiner, Huges]

Fourier Transform - Properties

- Fourier transform of the product of two functions is equivalent to the convolution of the individual Fourier transforms

  $$\mathfrak{F}\{f(x)g(x)\} = F(\omega) \otimes G(\omega)$$

- Convolution in the spatial domain is equivalent to multiplication in the frequency domain

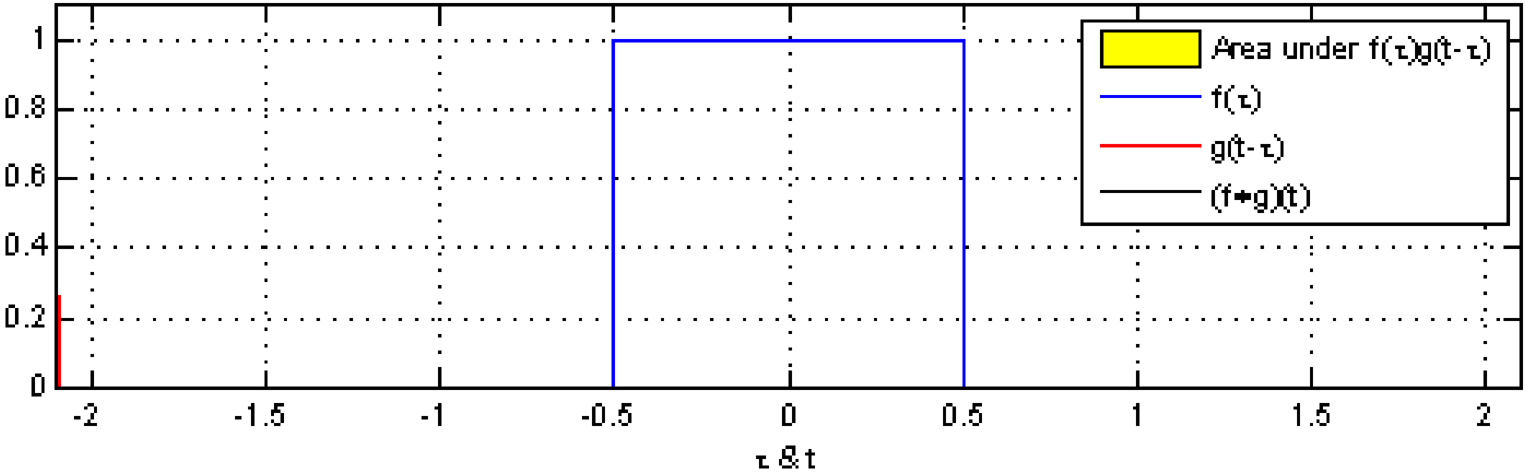
  $$\mathfrak{F}\{f(x) \otimes g(x)\} = F(\omega)G(\omega)$$

- Important in understanding how filtering and reconstruction affect the spectrum of a function

Convolution

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

– Convolution computes a weighted average of f using the weighting kernel g



[Wikipedia: Convolution]

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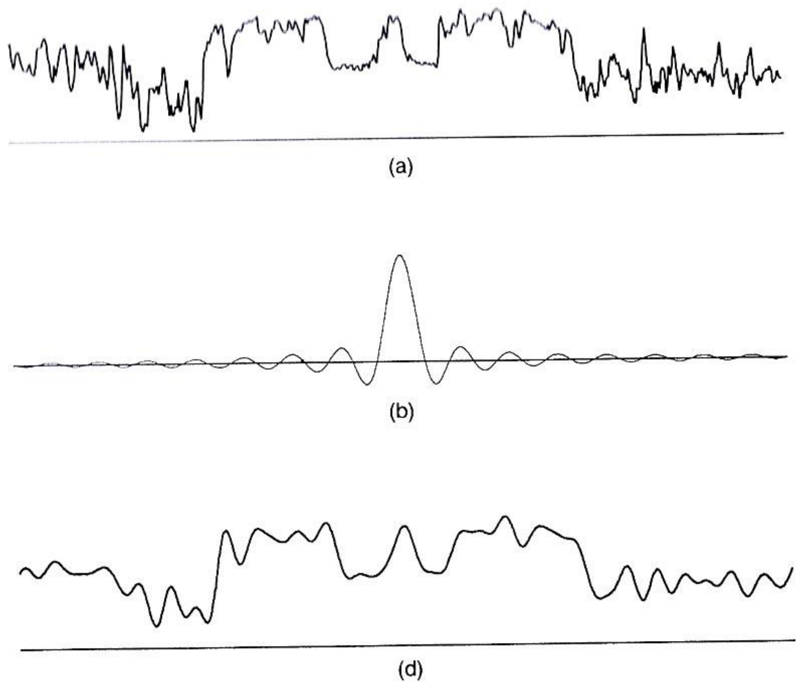
Low-Pass Filtering

– Convolution is used to filter and reconstruct functions



Original function

sinc function $f(x) = \frac{\sin(x)}{x}$

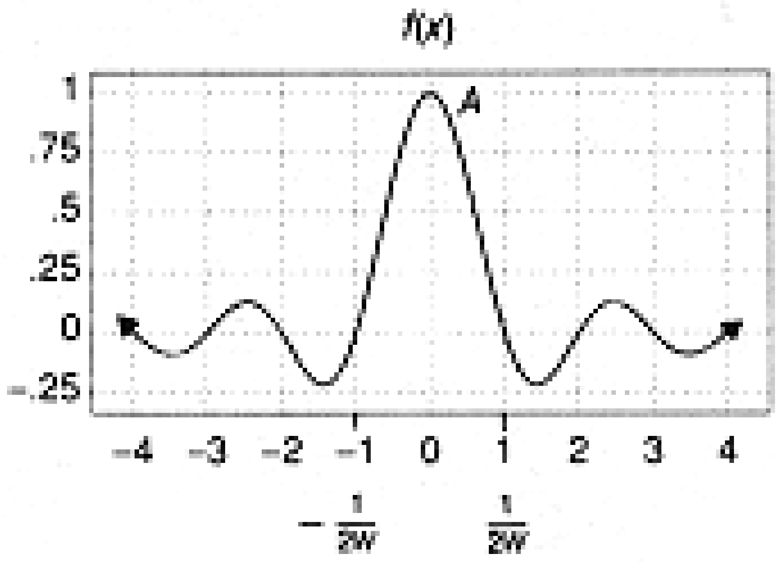
Band-limited signal

Fig. 14.23 Low-pass filtering in the spatial domain. (a) Original signal. (b) Sinc filter. (c) Signal with filter, with value of filtered signal shown as a black dot (●) at filter's origin. (d) Filtered signal. (Courtesy of George Wolberg, Columbia University.)

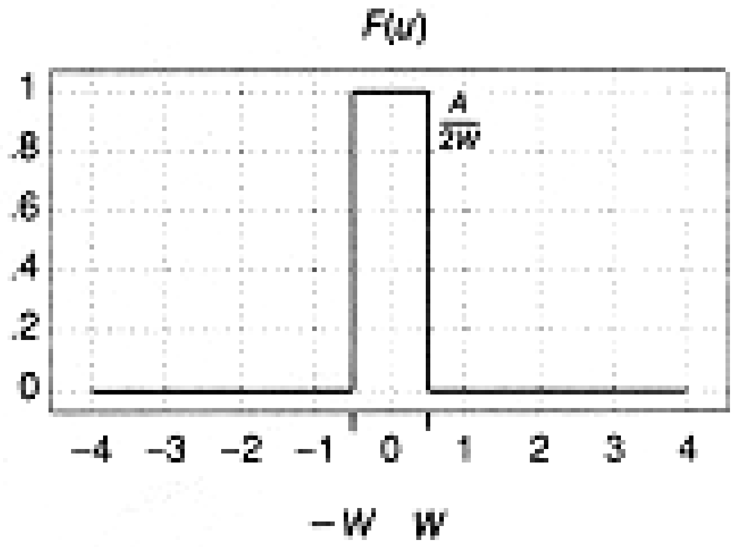
[Foley, van Dam, Feiner, Huges]

Low-Pass Filtering

– *sinc* function in spatial and frequency domain



sinc function

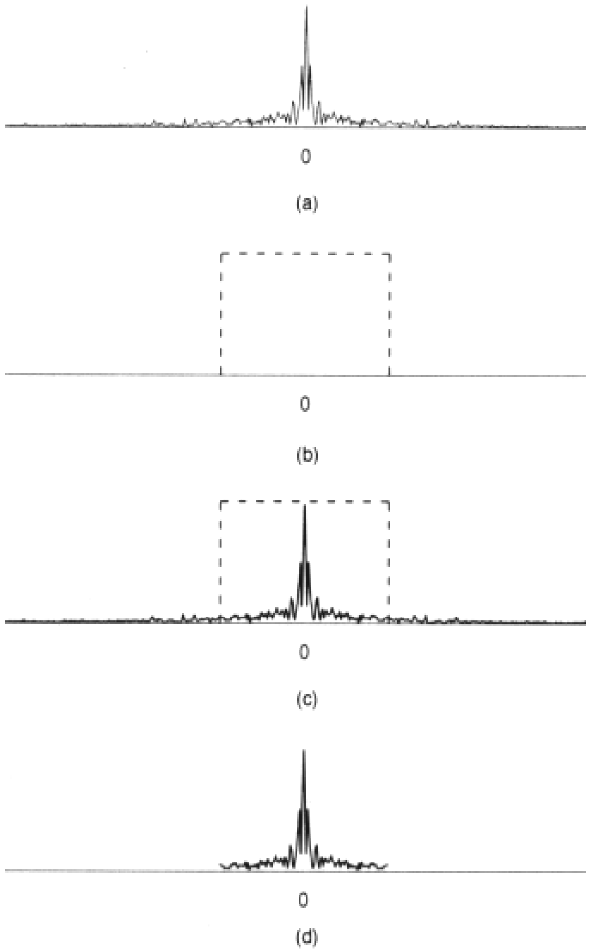


Box function

[Foley, van Dam, Feiner, Huges]

Low-Pass Filtering

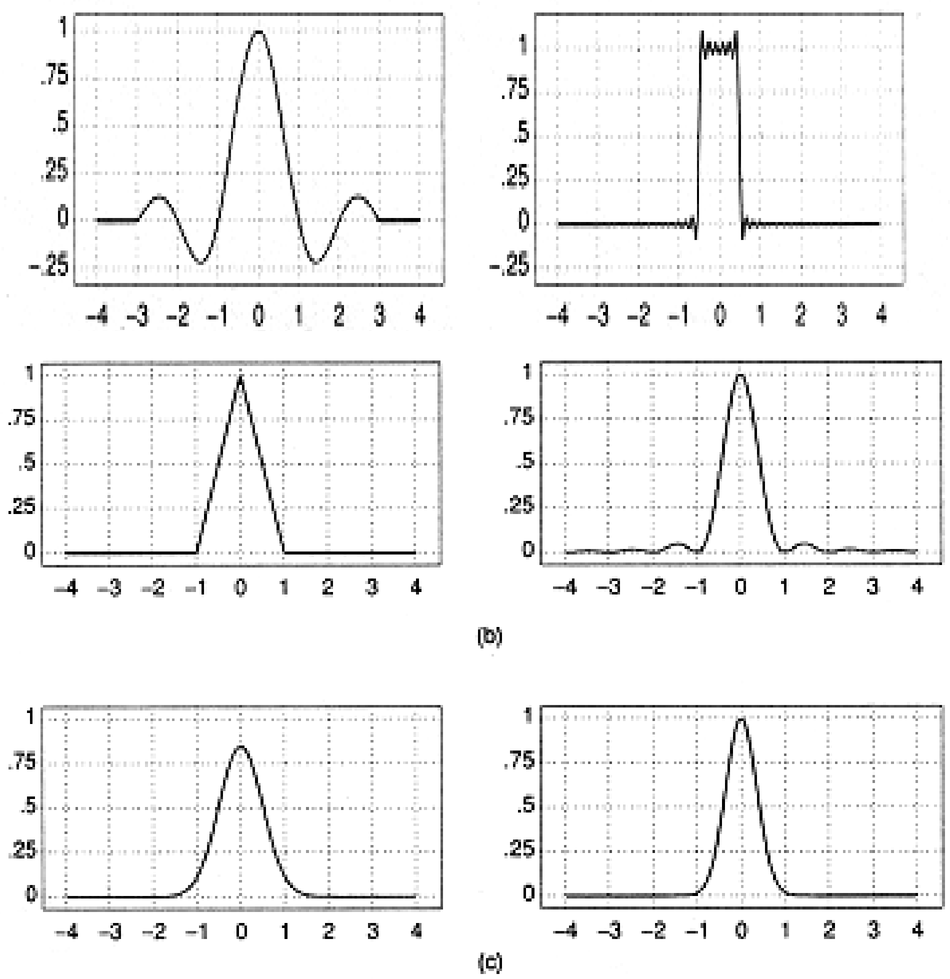
- Convolution with *sinc* function in the spatial domain corresponds to multiplication with a box function in the frequency domain
- Given a sampling rate, this low-pass filter completely suppresses all frequency components above the Nyquist frequency
  - Aliasing is avoided in the reconstruction process
- Applied in texturing
- In ray tracing, the original function cannot be simply filtered



[Foley, van Dam, Feiner, Huges]

Approximate Low-Pass Filtering

- Truncated *sinc*
  - Gibbs phenomenon
- Triangle
- Gaussian



[Foley, van Dam, Feiner, Huges]

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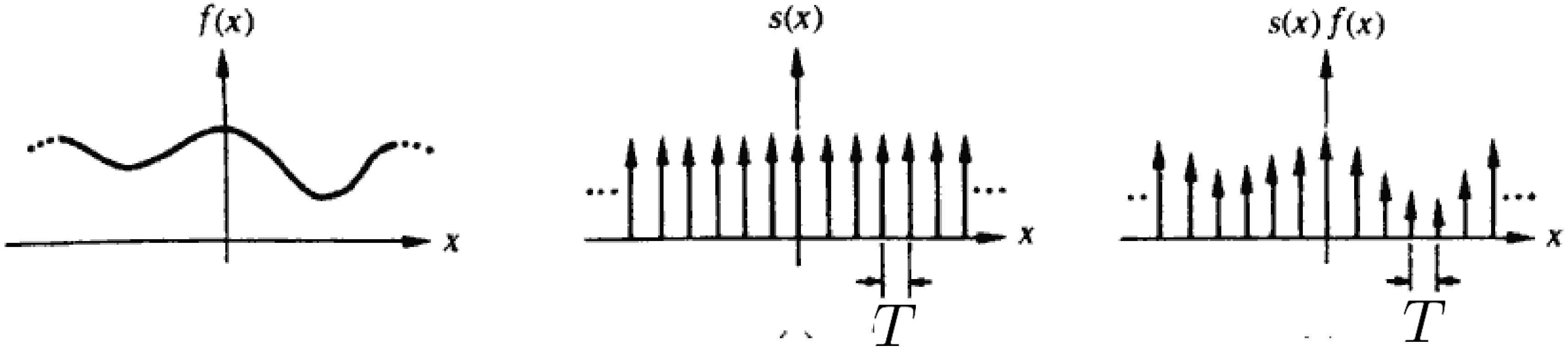
Sampling

– Sampling a function corresponds to multiplying it in the spatial domain by a Dirac comb function

$$\text{III}_T(x) = \sum_{i=-\infty}^{\infty} \delta(x - iT)$$



Original function

Dirac comb function (shah function)

Sampled function

[Foley, van Dam, Feiner, Huges]

Sampling

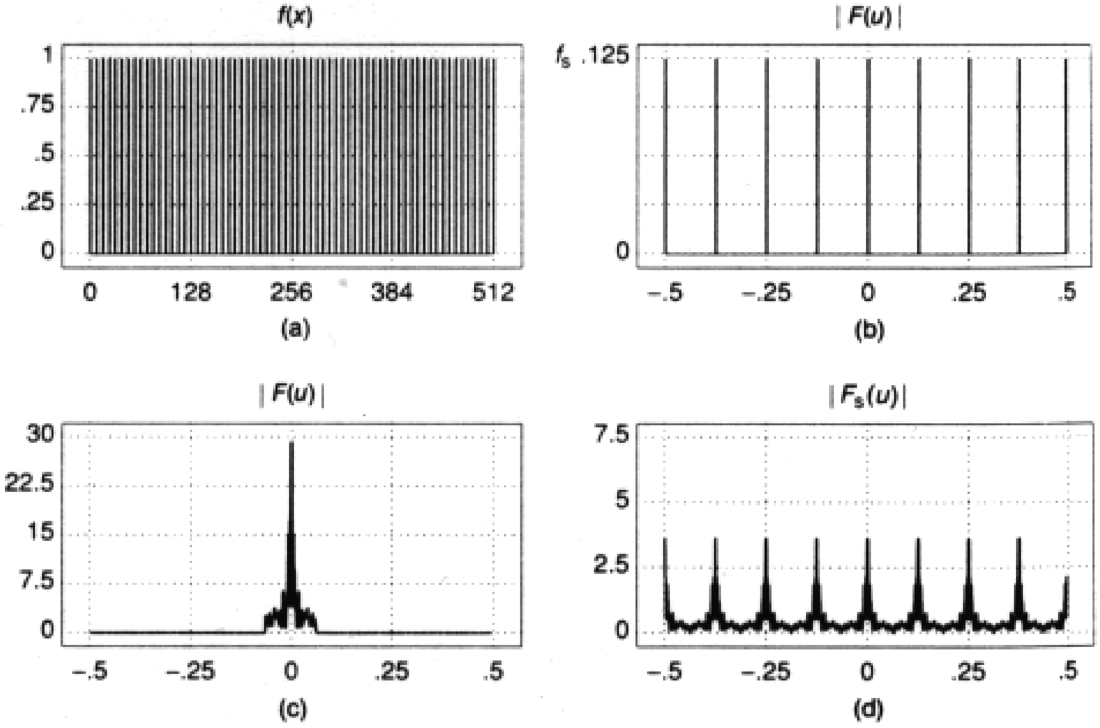
- In frequency domain, sampling is a convolution of the function's spectrum with a Dirac comb function

Dirac comb in the spatial domain
$\mathrm{III}_T(x)$

Dirac comb in the frequency domain
$\mathrm{III}_{1/T}(\omega)$

Spectrum of the function
$F(\omega)$

Spectrum of the sampled function
$F(\omega) \otimes \mathrm{III}_{1/T}(\omega)$



[Foley, van Dam, Feiner, Huges]

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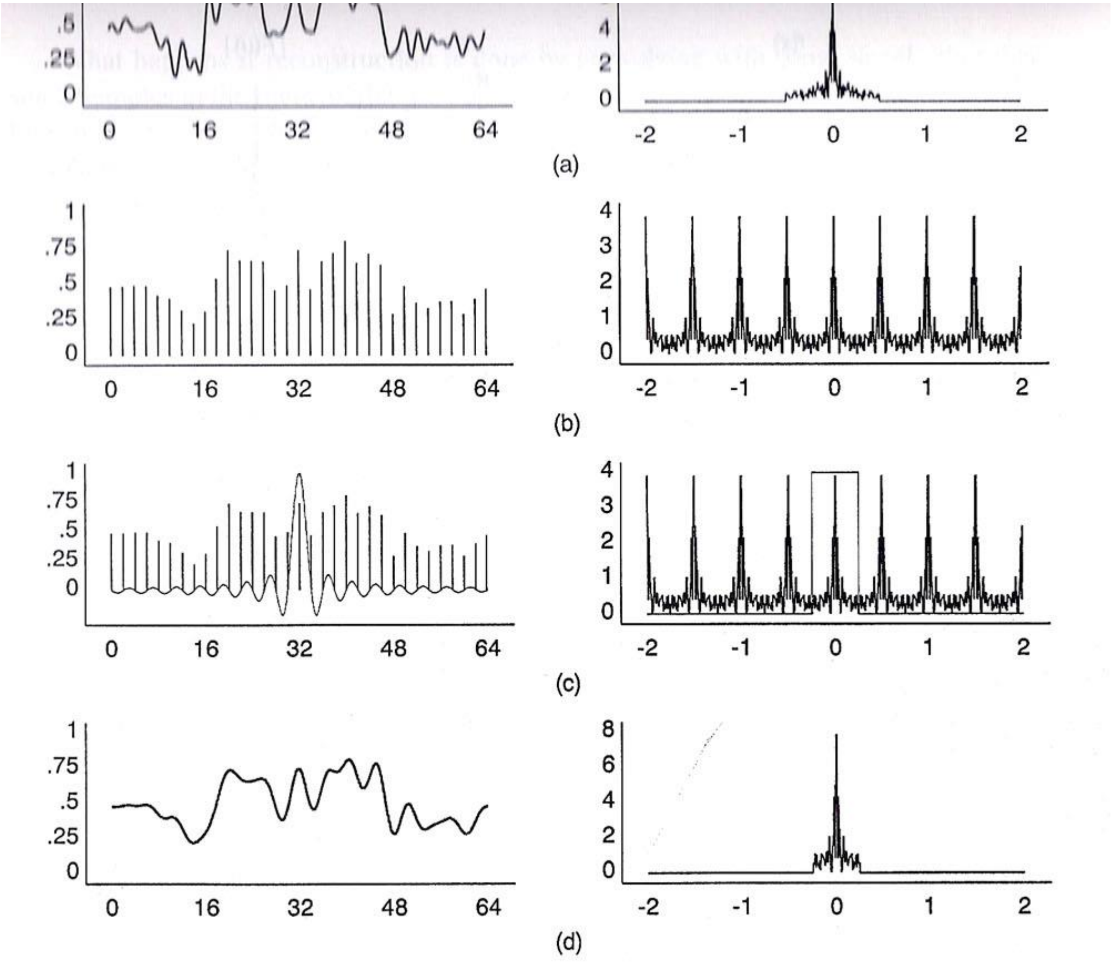
Reconstruction

- As a result of sampling, F contains an infinite number of replications of the spectrum of the original function f at multiples of the sampling frequency
- Reconstruction tries to remove all but the spectrum of the original function by multiplying with a box filter in the frequency domain (corresponding to a convolution of the sampled function with *sinc* in the spatial domain)
- Our visual system reconstructs a function from pixel values
- In ray tracing, incident radiance at pixels can be reconstructed from several samples

Reconstruction

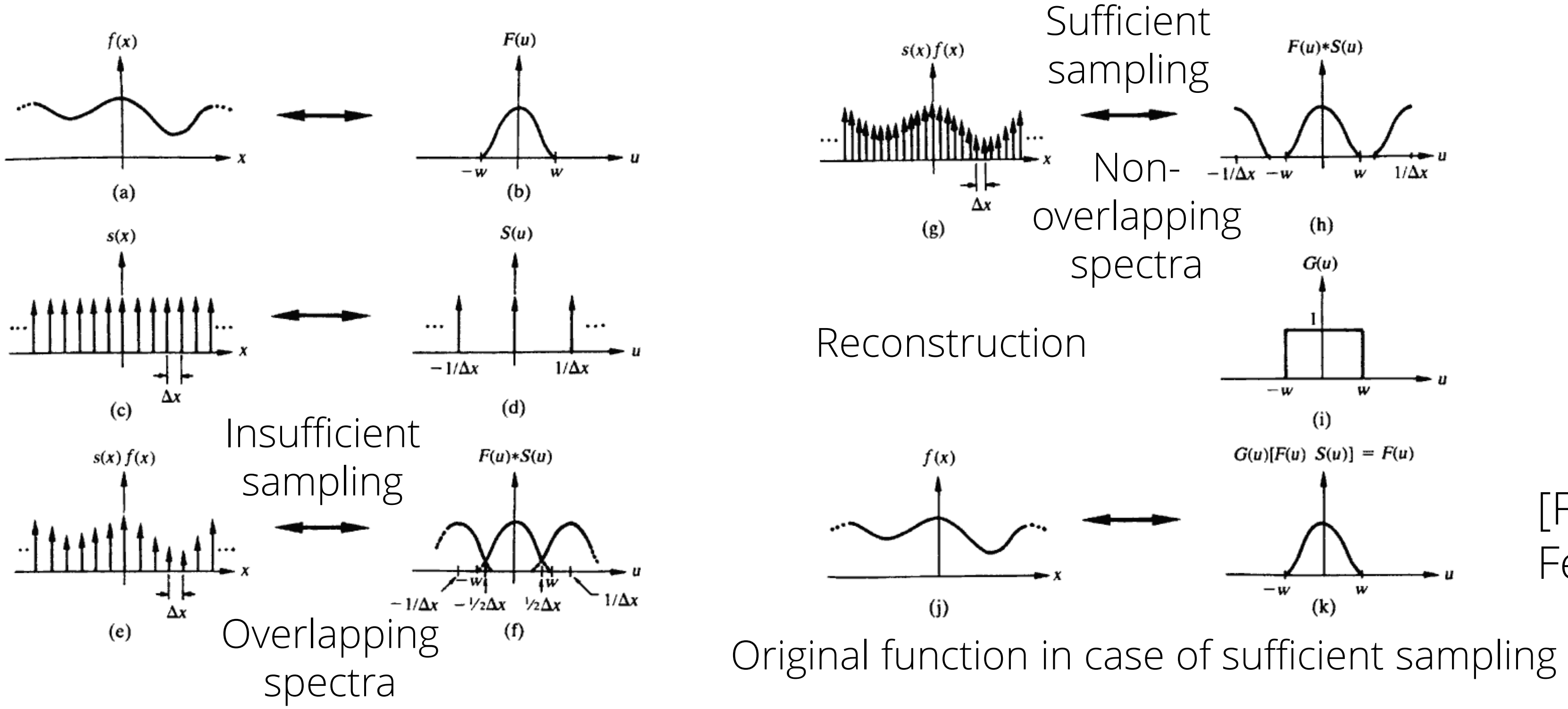
– Sampling and reconstruction in spatial and frequency domain



[Foley, van Dam, Feiner, Huges]

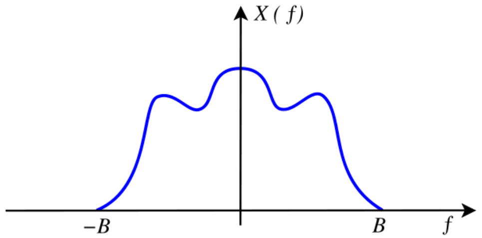
Aliasing

– If the sampling frequency is too low, the replicated copies of the spectra overlap and the spectrum of the original function cannot be reconstructed



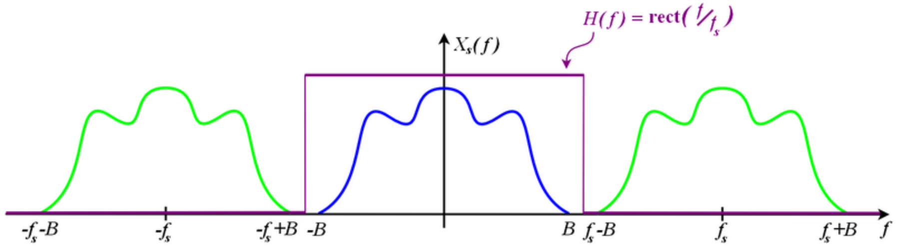
[Foley, van Dam, Feiner, Huges]

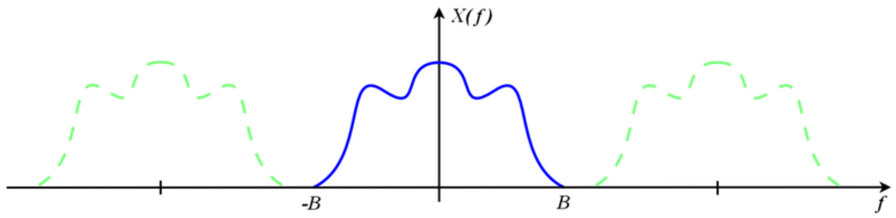
Aliasing



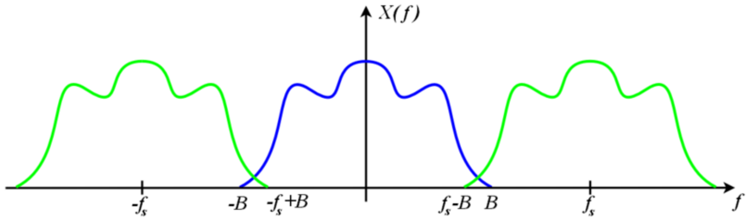
Spectrum of a band-limited function

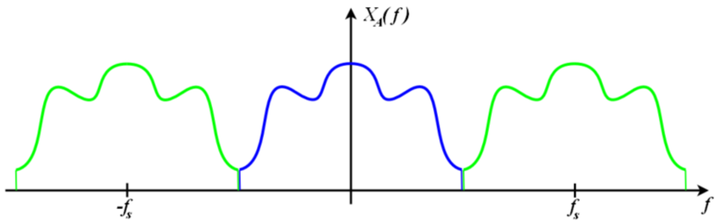
[Wikipedia: Nyquist-Shannon-Abtasttheorem]





Spectrum of sufficiently sampled band-limited function, a box filter can reconstruct the original spectrum





Spectrum of an insufficiently sampled band-limited function, the original spectrum cannot be reconstructed

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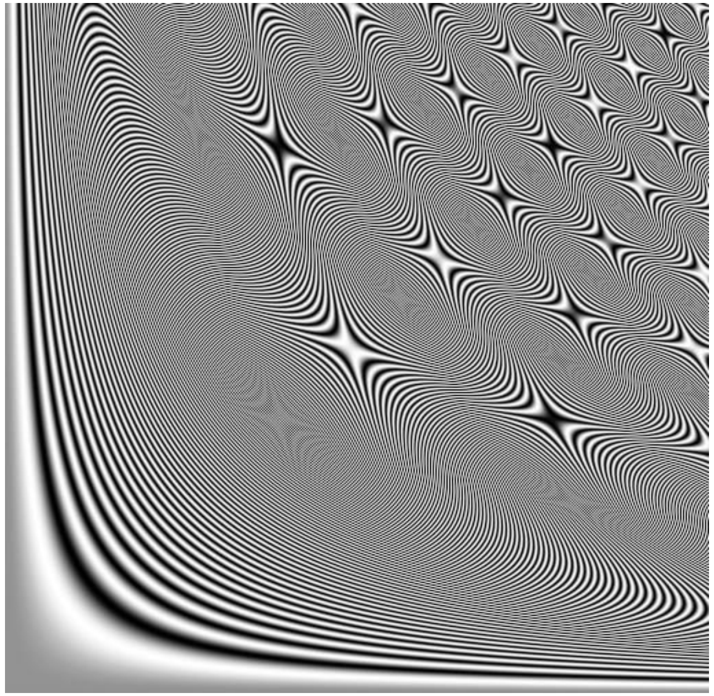
Antialiasing

- In texturing, textures are filtered according to the given sampling rate
  - prefiltering
- In ray tracing, the sampling rate and sampling patterns are adapted
  - Nonuniform sampling: tends to turn regular aliasing patterns into noise
  - Adaptive sampling: use more samples in case of large variations between adjacent samples (might still miss high frequencies, small details)
- In ray tracing, radiance at a pixel position is commonly reconstructed from samples within the pixel area and samples in adjacent pixels

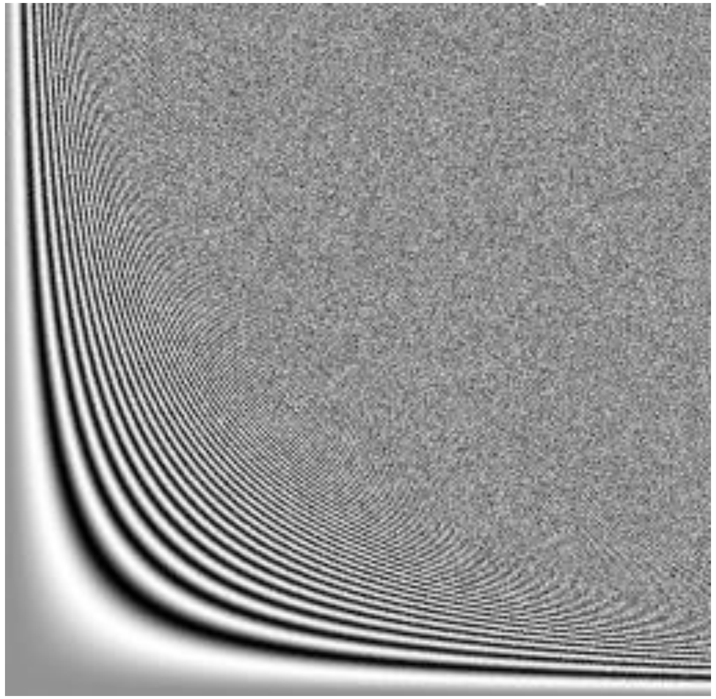
Antialiasing

$$f(x,y) = \frac{1}{2}(1+\sin(x^2y^2))$$



Regular sampling
with aliasing

[Suffern]



Non-uniform, random
sampling with noise

Summary

- Sampling of the continuous radiance function can cause aliasing
  - Moiré patterns
  - Jaggies
  - Missing details
- Fourier analysis helps to understand sampling, filtering / reconstruction, and aliasing effects
- Fourier transform converts between spatial and frequency domain
- Fourier transform of the product of two functions is equivalent to the convolution of the individual Fourier transforms