Computer Graphics
Particle Fluids

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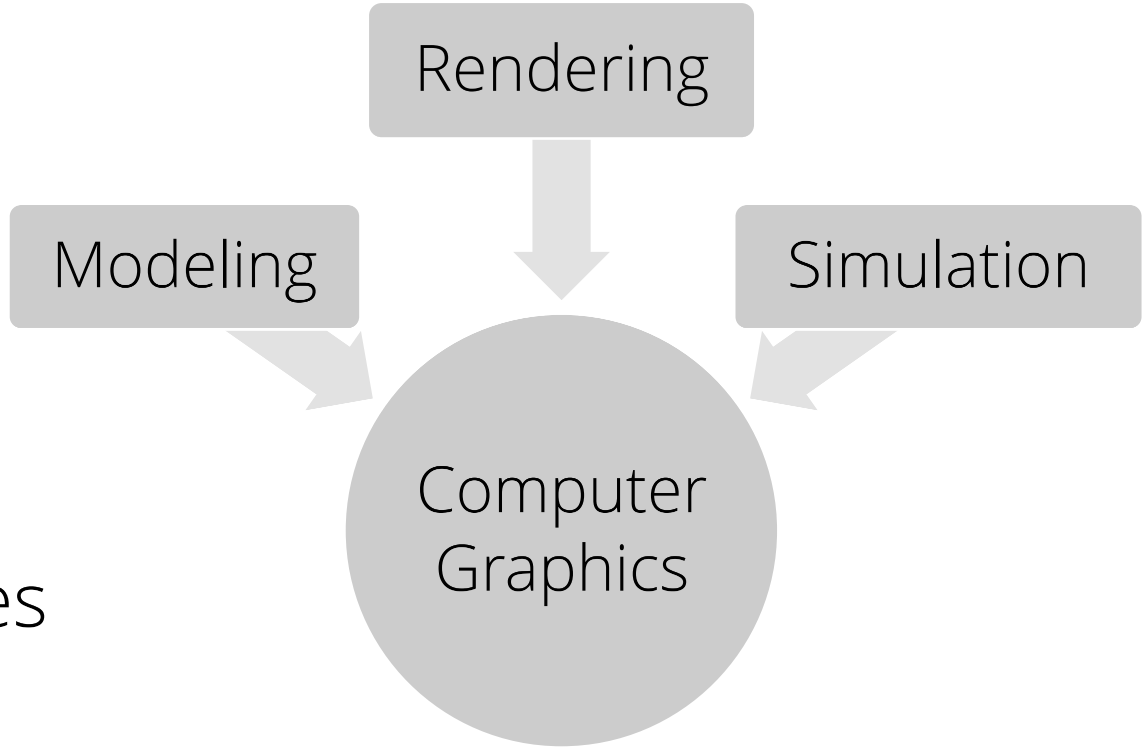
Course Topics

- Rendering
  - What is visible at a sensor?
    - Ray casting
    - Rasterization / Depth test
  - Which color does it have?
    - Phong
- Modeling
  - Parametric polynomial curves
- Simulation
  - Particle fluids

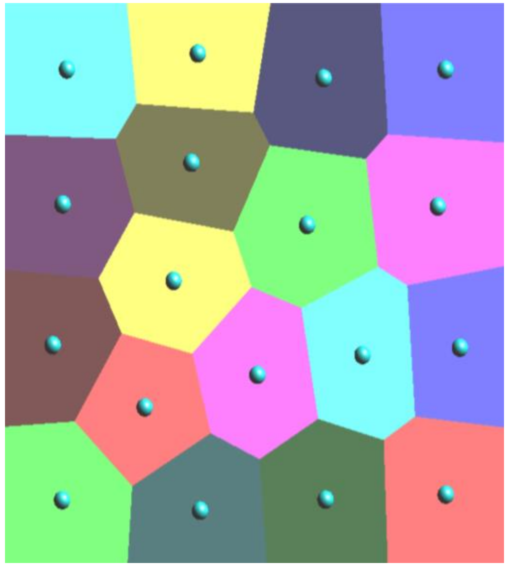


Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook

Particle Simulation

Fluid / Elastic object /
Rigid object



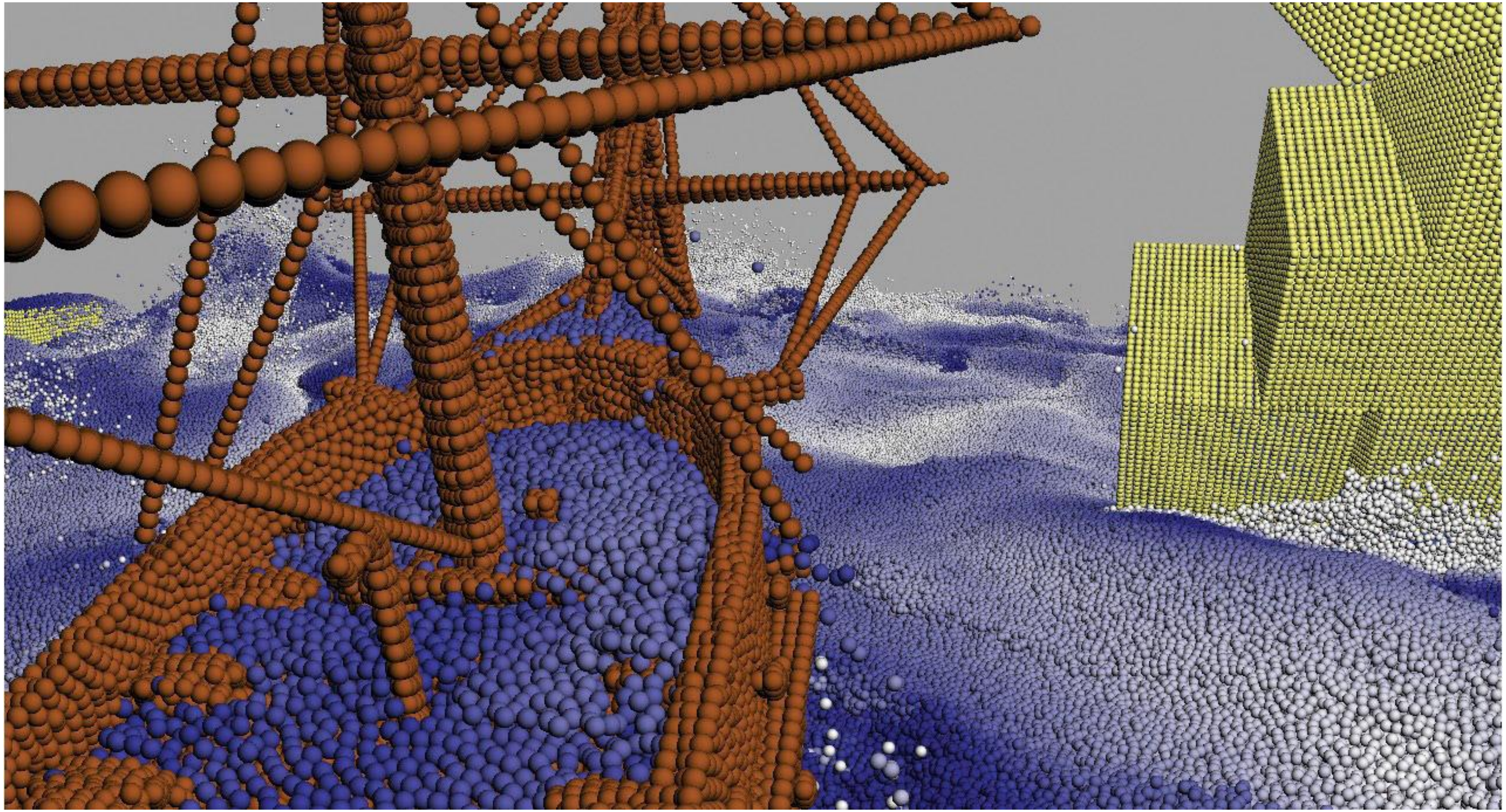
Set of parcels

$$\boldsymbol{x}_i^t$$

$$\boldsymbol{v}_i^t$$

Positions and
velocities of
parcels *i* over time *t*

Fluid and Solid Parcels



Simulation

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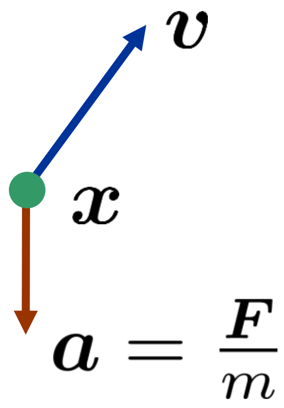
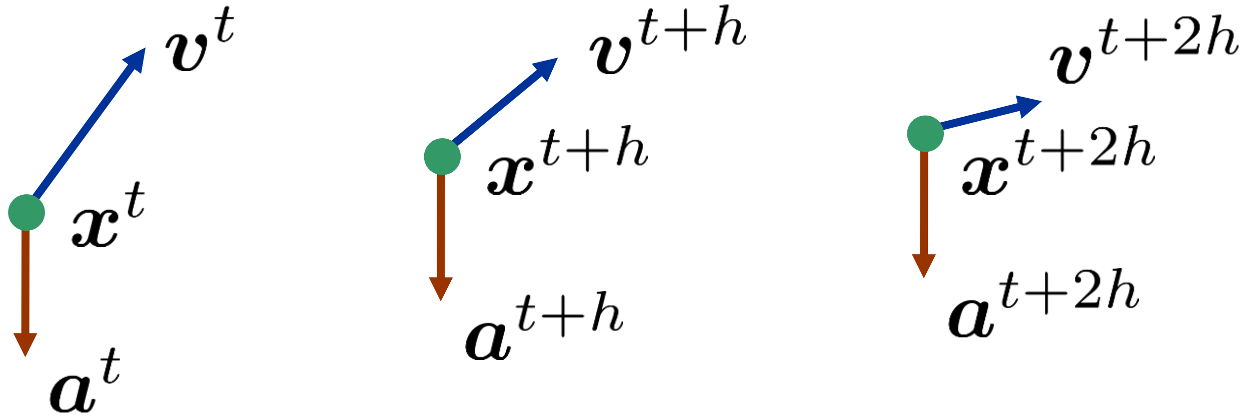
Particle Quantities

- Mass $m \in \mathbb{R}$
- Position $\boldsymbol{x} \in \mathbb{R}^3$
- Velocity $\boldsymbol{v} \in \mathbb{R}^3$
- Force $\boldsymbol{F} \in \mathbb{R}^3$
- Acceleration $\boldsymbol{a} = \frac{\boldsymbol{F}}{m} \in \mathbb{R}^3$



Time Discretization

– Quantities are considered at discrete time points $t, t+h$



h is the time step.

– Particle simulations are concerned with the computation of unknown future particle quantities $\boldsymbol{x}^{t+h}$, $\boldsymbol{v}^{t+h}$ from known current information $\boldsymbol{x}^t$, $\boldsymbol{v}^t$, $\boldsymbol{a}^t$
  – Where is the parcel? Which velocity does it have?

Governing Equations

- Newton's Second Law, motion equation

$$\frac{\mathrm{d}\boldsymbol{v}^t}{\mathrm{d}t} = \boldsymbol{a}^t = \frac{\boldsymbol{F}^t}{m} \qquad \frac{\mathrm{d}\boldsymbol{x}^t}{\mathrm{d}t} = \boldsymbol{v}^t$$

- Ordinary differential equations ODE
- Describe the behavior of $\boldsymbol{x}^t$ and $\boldsymbol{v}^t$ in terms of their time derivative
- Numerical integration is employed to approximatively solve the ODEs, i.e. to approximate the unknown functions $\boldsymbol{x}^t$ and $\boldsymbol{v}^t$

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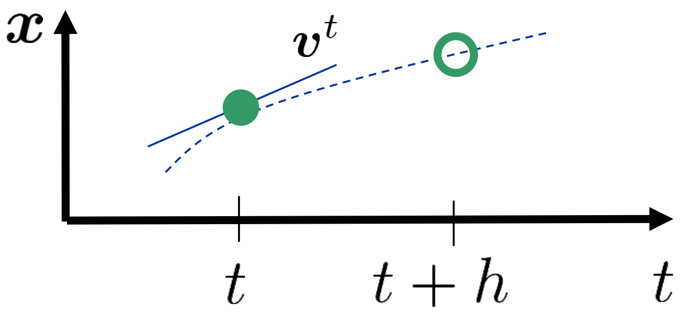
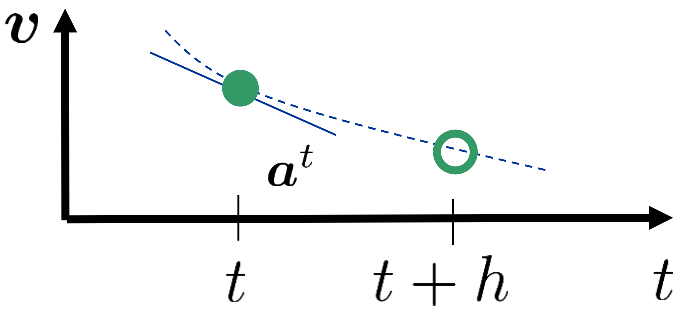
Initial Value Problem

- Functions $\boldsymbol{x}^t$ and $\boldsymbol{v}^t$ represent the particle motion
- Initial values $\boldsymbol{x}^t$ and $\boldsymbol{v}^t$ are given
- First-order differential equations are given

$$\frac{\mathrm{d}\boldsymbol{x}^t}{\mathrm{d}t} = \boldsymbol{v}^t \quad \frac{\mathrm{d}\boldsymbol{v}^t}{\mathrm{d}t} = \boldsymbol{a}^t$$

- How to estimate $\boldsymbol{x}^{t+h}$ and $\boldsymbol{v}^{t+h}$?





Explicit Euler

- Governing equations

$$\frac{\mathrm{d}\boldsymbol{x}^t}{\mathrm{d}t} = \boldsymbol{v}^t \quad \frac{\mathrm{d}\boldsymbol{v}^t}{\mathrm{d}t} = \boldsymbol{a}^t$$

- Initialization $\boldsymbol{x}^t = \boldsymbol{x}^{\text{init}}$, $\boldsymbol{v}^t = \boldsymbol{v}^{\text{init}}$, $\boldsymbol{a}^t$, h
- Explicit Euler update

$$\boldsymbol{x}^{t+h} = \boldsymbol{x}^t + h\frac{\mathrm{d}\boldsymbol{x}^t}{\mathrm{d}t} + O(h^2) = \boldsymbol{x}^t + h\boldsymbol{v}^t + O(h^2)$$

$$\boldsymbol{v}^{t+h} = \boldsymbol{v}^t + h\frac{\mathrm{d}\boldsymbol{v}^t}{\mathrm{d}t} + O(h^2) = \boldsymbol{v}^t + h\boldsymbol{a}^t + O(h^2)$$

Taylor approximation

Alternative Updates, e.g. Verlet

- Taylor approximations of $\boldsymbol{x}^{t+h}$ and $\boldsymbol{x}^{t-h}$

$$\boldsymbol{x}^{t+h} = \boldsymbol{x}^t + h\boldsymbol{v}^t + \frac{h^2}{2}\boldsymbol{a}^t + \frac{h^3}{6}\frac{\mathrm{d}^3\boldsymbol{x}^t}{\mathrm{d}t^3} + O(h^4)$$

$$\boldsymbol{x}^{t-h} = \boldsymbol{x}^t - h\boldsymbol{v}^t + \frac{h^2}{2}\boldsymbol{a}^t - \frac{h^3}{6}\frac{\mathrm{d}^3\boldsymbol{x}^t}{\mathrm{d}t^3} + O(h^4)$$

- Adding both approximations leads to the position update

$$\boldsymbol{x}^{t+h} = 2\boldsymbol{x}^t - \boldsymbol{x}^{t-h} + h^2\boldsymbol{a}^t + O(h^4)$$

- Velocity update, e.g.

$$\boldsymbol{v}^{t+h} = \frac{\boldsymbol{x}^{t+h} - \boldsymbol{x}^t}{h} + O(h)$$

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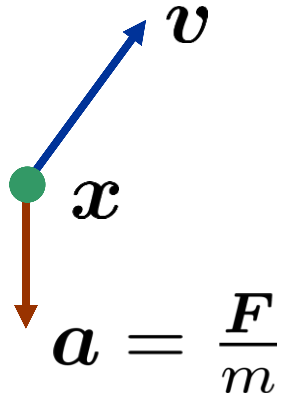
- Acceleration $\boldsymbol{a} = \frac{\boldsymbol{F}}{m} \in \mathbb{R}^3$
- Density $\rho \in \mathbb{R}$
- Pressure $p \in \mathbb{R}$



Governing Equations for a Fluid

- Particle positions $\boldsymbol{x}_i^t$ and the respective attributes are advected with the local fluid velocity $\boldsymbol{v}_i^t$
  $$\frac{\mathrm{d}\boldsymbol{x}_i^t}{\mathrm{d}t} = \boldsymbol{v}_i^t$$
- Time rate of change of the velocity $\boldsymbol{v}_i^t$ of an advected sample is governed by the Lagrange form of the Navier-Stokes equation
  $$\frac{\mathrm{d}\boldsymbol{v}_i^t}{\mathrm{d}t} = -\frac{1}{\rho_i^t}\nabla p_i^t + \nu\nabla^2\boldsymbol{v}_i^t + \frac{\boldsymbol{F}_i^{t,\mathrm{other}}}{m_i}$$

Accelerations

Accelerations in a Fluid

- $-\frac{1}{\rho_i^t}\nabla p_i^t$: Acceleration due to pressure differences
  - Pressure is proportional to compression
  - Particle are accelerated from areas with high pressure / compression to areas with lower pressure / compression
  - Small and preferably constant density deviations / compressions are important for high-quality simulations

Accelerations in a Fluid

$\nu \nabla^2 \boldsymbol{v}_i^t$: Acceleration due to friction forces between particles with different velocities

- Minimizes the difference between a particle velocity and the average velocity of all adjacent particles

$\frac{\boldsymbol{F}_i^{t,\text{other}}}{m_i}$: E.g., gravity

Acceleration Terms – 3D

– Incompressibility

$$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\begin{pmatrix} \frac{\partial p}{\partial x_x} \\ \frac{\partial p}{\partial x_y} \\ \frac{\partial p}{\partial x_z} \end{pmatrix}$$

– Viscosity

$$\nu\nabla^2\boldsymbol{v} = \nu\nabla\cdot(\nabla\boldsymbol{v}) = \nu\nabla\cdot\begin{pmatrix} \frac{\partial v_x}{\partial x_x} & \frac{\partial v_x}{\partial x_y} & \frac{\partial v_x}{\partial x_z} \\ \frac{\partial v_y}{\partial x_x} & \frac{\partial v_y}{\partial x_y} & \frac{\partial v_y}{\partial x_z} \\ \frac{\partial v_z}{\partial x_x} & \frac{\partial v_z}{\partial x_y} & \frac{\partial v_z}{\partial x_z} \end{pmatrix} = \nu\begin{pmatrix} \frac{\partial^2 v_x}{\partial x_x^2} + \frac{\partial^2 v_x}{\partial x_y^2} + \frac{\partial^2 v_x}{\partial x_z^2} \\ \frac{\partial^2 v_y}{\partial x_x^2} + \frac{\partial^2 v_y}{\partial x_y^2} + \frac{\partial^2 v_y}{\partial x_z^2} \\ \frac{\partial^2 v_z}{\partial x_x^2} + \frac{\partial^2 v_z}{\partial x_y^2} + \frac{\partial^2 v_z}{\partial x_z^2} \end{pmatrix}$$

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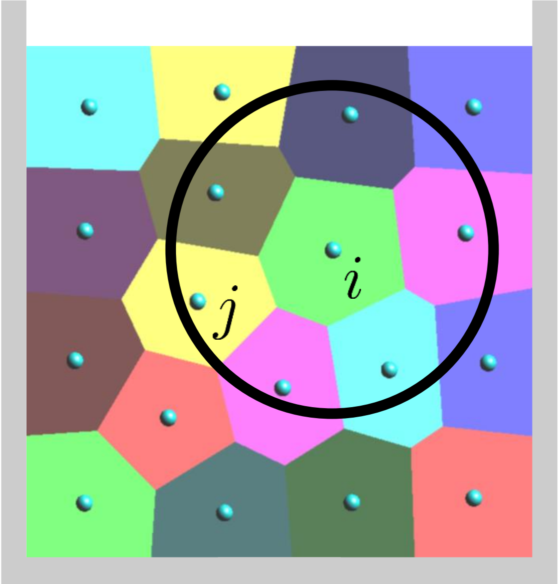
Smoothed Particle Hydrodynamics SPH

- Interpolates quantities at arbitrary positions and approximates the spatial derivatives with a finite number of samples, i.e. adjacent particles
- SPH in a fluid simulation
  - Fluid is represented with particles
  - Particle positions and velocities are governed by $\frac{\mathrm{d}\boldsymbol{x}_i^t}{\mathrm{d}t} = \boldsymbol{v}_i^t$ and $\frac{\mathrm{d}\boldsymbol{v}_i^t}{\mathrm{d}t} = -\frac{1}{\rho_i^t}\nabla p_i^t + \nu\nabla^2\boldsymbol{v}_i^t + \frac{\boldsymbol{F}_i^{t,\mathrm{other}}}{m_i}$
  - ρ_i^t, $-\frac{1}{\rho_i^t}\nabla p_i^t$, $\nu\nabla^2\boldsymbol{v}_i^t$ and $\frac{\boldsymbol{F}_i^{t,\mathrm{other}}}{m_i}$ are computed with SPH

SPH Interpolation

– Quantity A_i at an arbitrary position $\boldsymbol{x}_i$ is approximately computed with a set of known quantities A_j at sample positions $\boldsymbol{x}_j$:
$A_i = \sum_j A_j \frac{m_j}{\rho_j} W_{ij}$

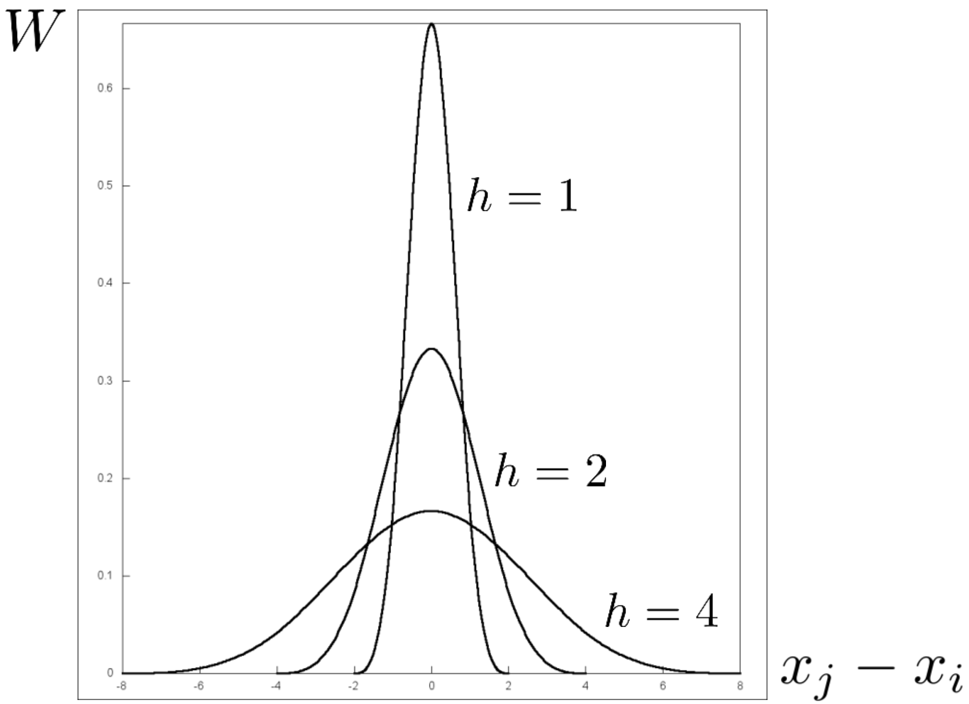


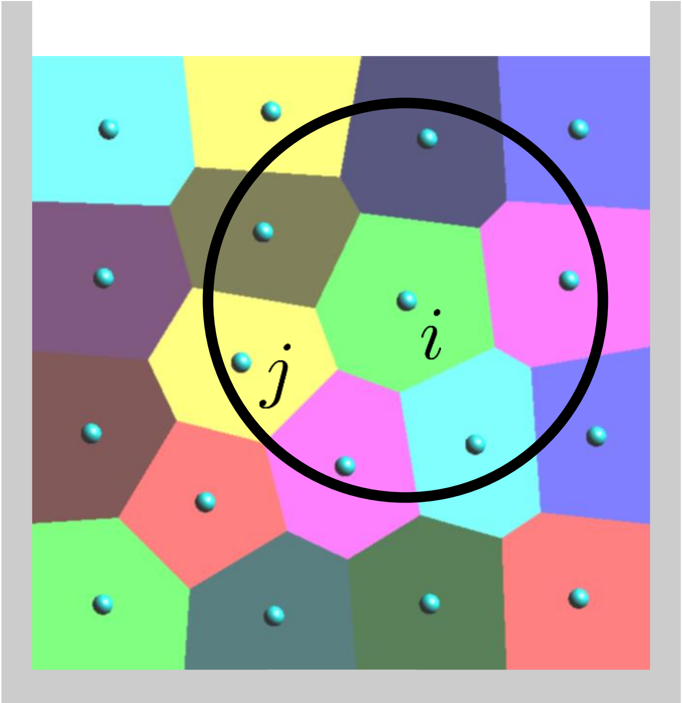
– W_{ij} is a kernel function that weights the contributions of sample positions $\boldsymbol{x}_j$ their distance to $\boldsymbol{x}_i$

– Spatial derivatives:

$\nabla A_i = \sum_j A_j \frac{m_j}{\rho_j} \nabla W_{ij}$

SPH Interpolation - Illustrations





$$W_{ij} = W(x_j - x_i) = \frac{1}{6h} \begin{cases} \left(2 - \frac{\|x_j - x_i\|}{h}\right)^3 - 4\left(1 - \frac{\|x_j - x_i\|}{h}\right)^3 & 0 \leq \frac{\|x_j - x_i\|}{h} < 1 \\ \left(2 - \frac{\|x_j - x_i\|}{h}\right)^3 & 1 \leq \frac{\|x_j - x_i\|}{h} < 2 \\ 0 & \frac{\|x_j - x_i\|}{h} \geq 2 \end{cases}$$

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Density

– Explicit SPH form

$$\rho_i = \sum_j \rho_j \frac{m_j}{\rho_j} W_{ij} = \sum_j m_j W_{ij}$$

Pressure

- Quantifies fluid compression
  - E.g., state equation $p_i = \max\left(k(\frac{\rho_i}{\rho_0} - 1), 0\right)$
  - Rest density of the fluid ρ_0
  - User-defined stiffness k
- Pressure acceleration with SPH
  - $\boldsymbol{a}_i^{\mathrm{p}} = -\frac{1}{\rho_i}\nabla p_i = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla W_{ij}$
  - Accelerates particles from high to low pressure, i.e. from high to low compression to minimize density deviation $\frac{\rho_i}{\rho_0} - 1$

Pressure values in SPH implementations should always be non-negative.

SPH Discretizations

- Density computation $\rho_i = \sum_j m_j W_{ij}$
- Pressure acceleration $-\frac{1}{\rho_i}\nabla p_i = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2}\right)\nabla W_{ij}$
- Viscosity acceleration $\nu \nabla^2 \boldsymbol{v}_i = 2\nu \sum_j \frac{m_j}{\rho_j} \frac{\boldsymbol{v}_{ij} \cdot \boldsymbol{x}_{ij}}{\boldsymbol{x}_{ij} \cdot \boldsymbol{x}_{ij} + 0.01h^2} \nabla W_{ij}$

Simple SPH Fluid Solver

for all *particle* i **do**	
find neighbors j	Compute adjacent particles for SPH sums
for all *particle* i **do**	
$\rho_i = \sum_j m_j W_{ij}$	Compute density
$p_i = k(\frac{\rho_i}{\rho_0} - 1)$	Compute pressure
for all *particle* i **do**	
$\boldsymbol{a}_i^{\text{nonp}} = \nu \nabla^2 \boldsymbol{v}_i + \boldsymbol{g}$	Compute non-pressure accelerations
$\boldsymbol{a}_i^{\text{p}} = -\frac{1}{\rho_i} \nabla p_i$	Compute pressure acceleration
$\boldsymbol{a}_i^t = \boldsymbol{a}_i^{\text{nonp}} + \boldsymbol{a}_i^{\text{p}}$	
for all *particle* i **do**	
$\boldsymbol{v}_i^{t+\Delta t} = \boldsymbol{v}_i^t + \Delta t \boldsymbol{a}_i^t$	Explicit Euler for velocity update
$\boldsymbol{x}_i^{t+\Delta t} = \boldsymbol{x}_i^t + \Delta t \boldsymbol{v}_i^{t+\Delta t}$	Implicit Euler for position update

Summary

- Fluid is subdivided into particles
- Navier-Stokes equation states particle accelerations
- SPH states how to approximate these accelerations using adjacent particles (space discretization)
- Fluid solver
  - Compute accelerations
  - Update positions and velocities (time discretization)
  - Accelerations require neighbor search for SPH approximations and density deviation for pressure acceleration

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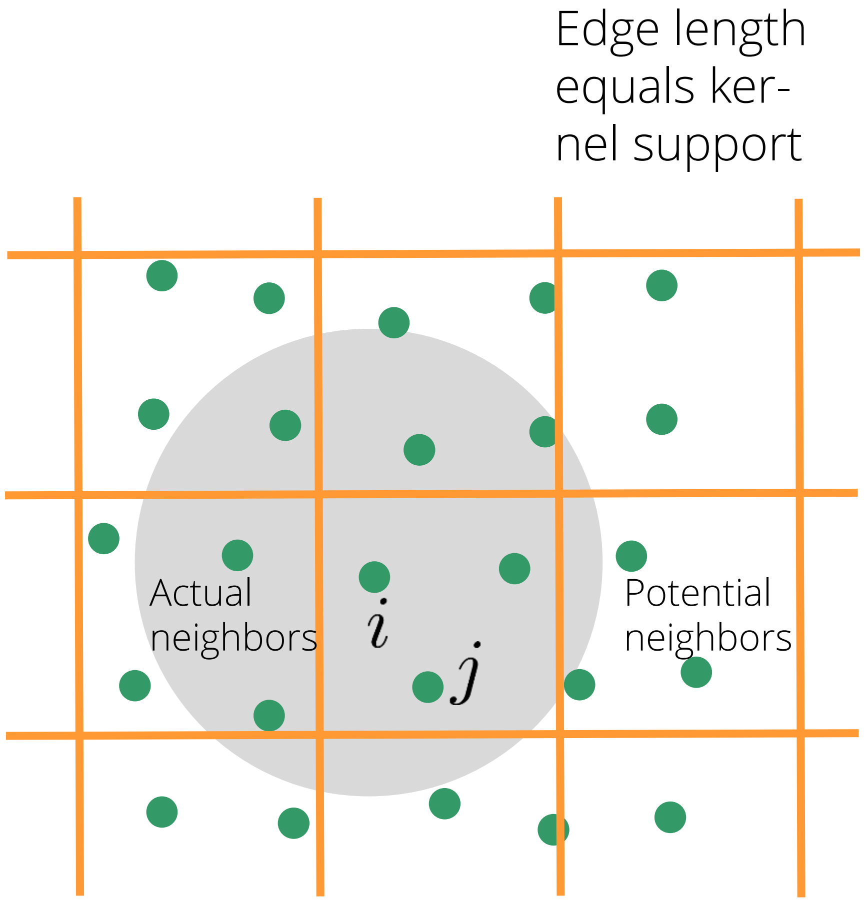
Uniform Grid - Concept

- Particles are stored in cells
- In *d*-D, potential neighbors in 3^d cells are queried to estimate actual neighbors
- Cell size equals the kernel support of a particle
  - Larger cells increase the number of tested particles
  - Smaller cells increase the number of tested cells



Uniform Grid - Implementation

- Compute unique cell identifier per particle
  - Space-filling curves
- Sort particles with respect to cell identifier
  - Particles in the same cell are close to each other
- Map cells to a hash table
  - No explicit representation of the uniform grid
  - Infinitely large grids can be handled
- See Simulation in Computer Graphics

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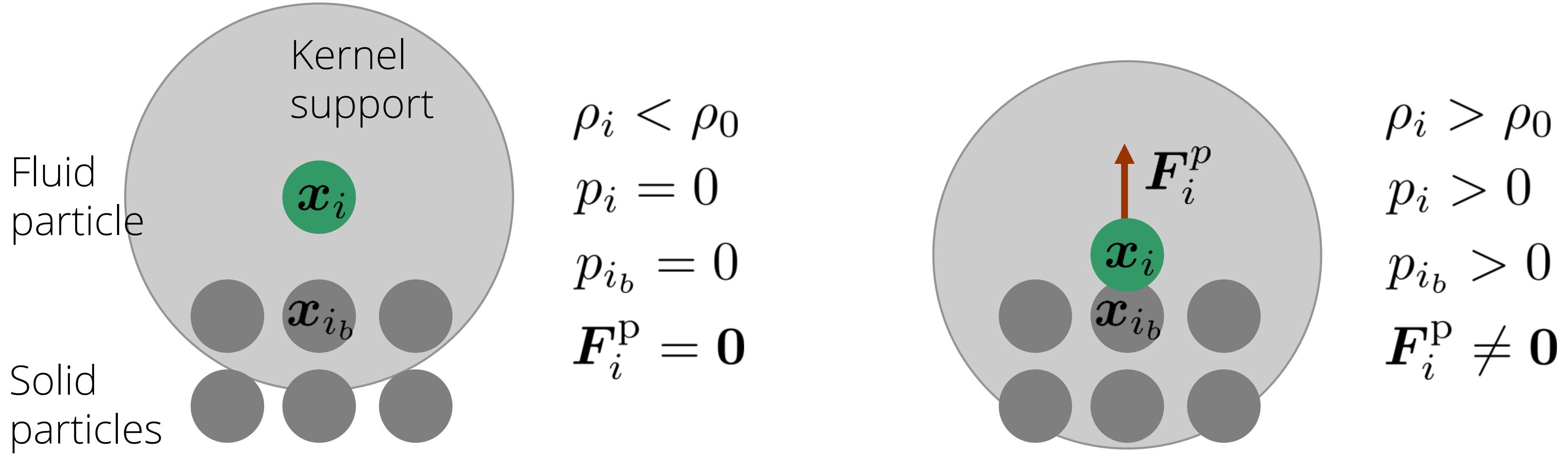
Concept

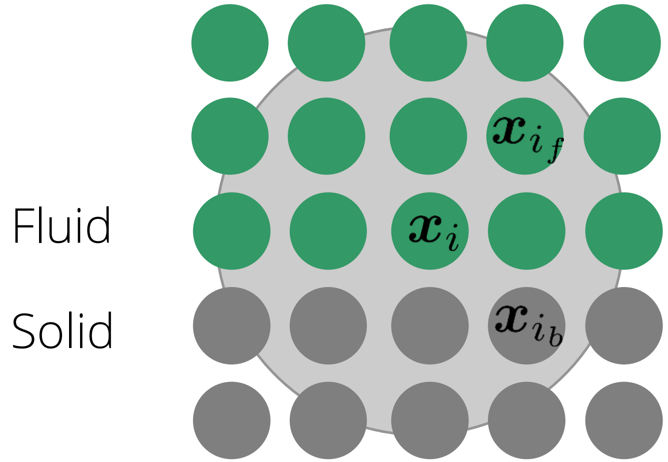
– Boundaries are sampled with particles that contribute to density, pressure and pressure acceleration



– Boundary handling: How to compute $\rho_i, p_i, p_{i_b}, \boldsymbol{F}_i^{\mathrm{p}}$?

Several Layers with Uniform Boundary Samples

– Boundary particles are handled as static fluid samples



$$\rho_i = \sum_{i_f} m_{i_f} W_{ii_f} + \sum_{i_b} m_{i_b} W_{ii_b}$$

Boundary neighbors contribute to the density

$$m_i = m_{i_f} = m_{i_b}$$

All samples have the same size, i.e. same mass and rest density

$$\rho_i = m_i \sum_{i_f} W_{ii_f} + m_i \sum_{i_b} W_{ii_b}$$

$$p_i = k\left(\frac{\rho_i}{\rho_0} - 1\right)$$

– Pressure acceleration

$$\boldsymbol{a}_i^{\mathrm{p}} = \underbrace{-m_i \sum_{i_f} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_f}}{\rho_{i_f}^2}\right) \nabla W_{ii_f}}_{\text{Contributions from fluid neighbors}} \underbrace{- m_i \sum_{i_b} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_b}}{\rho_{i_b}^2}\right) \nabla W_{ii_b}}_{\text{Contributions from boundary neighbors}}$$

All samples have the same size, i.e. same mass and rest density

Pressure at Boundary Samples

- Pressure acceleration at boundaries requires pressure at boundary samples
- Various solutions, e.g. mirroring, extrapolation
- Mirroring
  - Formulation with unknown boundary pressure p_{i_b}

    $$\boldsymbol{a}_i^{\mathrm{p}} = -m_i \sum_{i_f} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_f}}{\rho_{i_f}^2} \right) \nabla W_{i i_f} - m_i \sum_{i_b} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_b}}{\rho_{i_b}^2} \right) \nabla W_{i i_b}$$

  - Mirroring of pressure and density from fluid to boundary $p_{i_b} = p_i$

    $$\boldsymbol{a}_i^{\mathrm{p}} = -m_i \sum_{i_f} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_f}}{\rho_{i_f}^2} \right) \nabla W_{i i_f} - m_i \sum_{i_b} \left(\frac{p_i}{\rho_i^2} + \frac{p_i}{\rho_i^2} \right) \nabla W_{i i_b}$$

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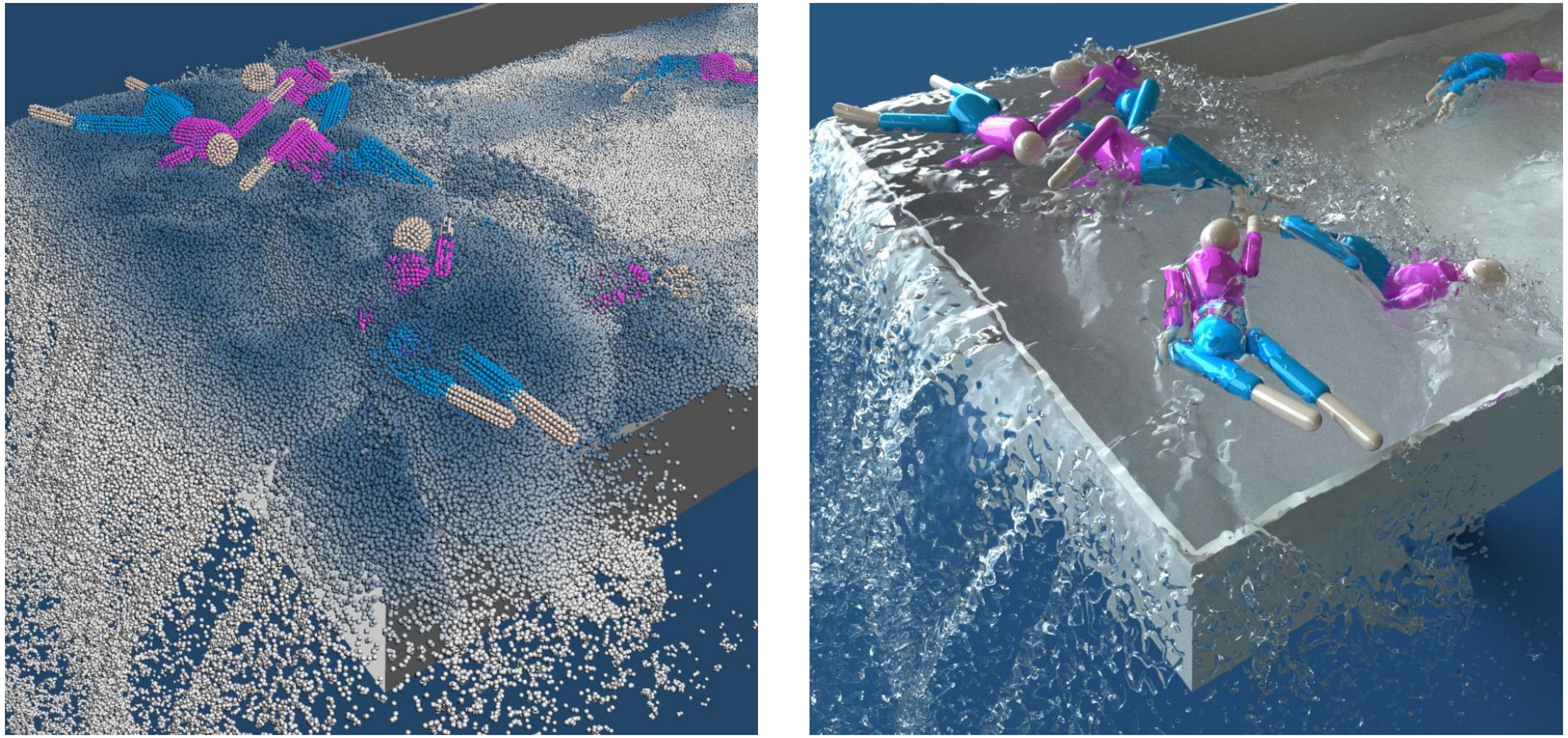
– Reconstruction and rendering of a triangulated iso-surface



Akinci et al., ACM Transactions on Graphics, 2012

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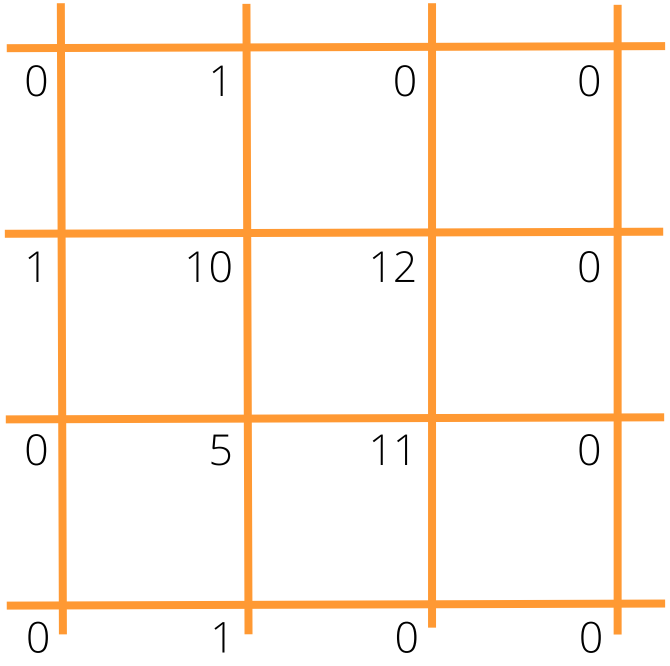
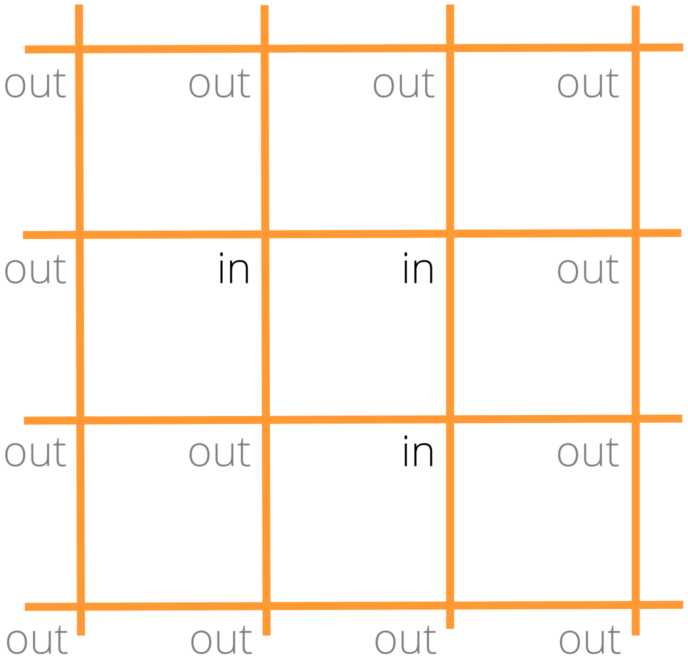
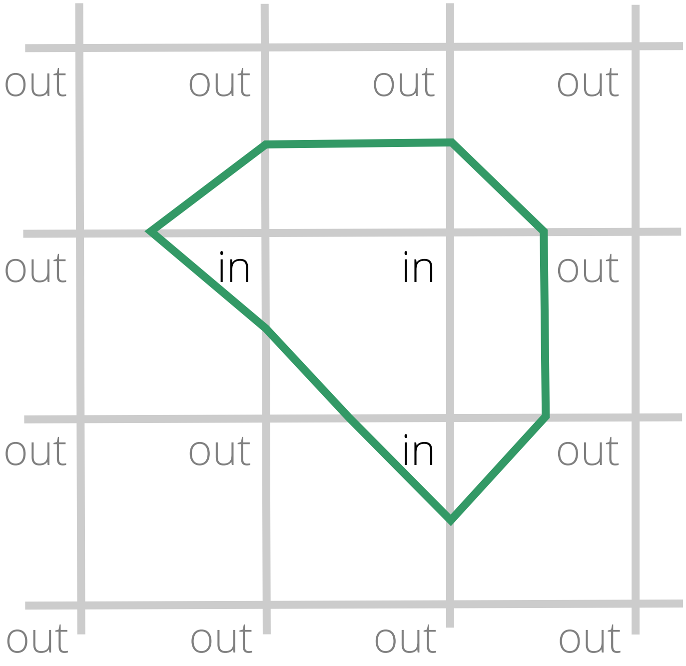
Iso-Surface Reconstruction – Marching Cubes



Input: Scalar field



Classification with respect to an iso-value, e.g. 8



Output: Triangulated iso-surface

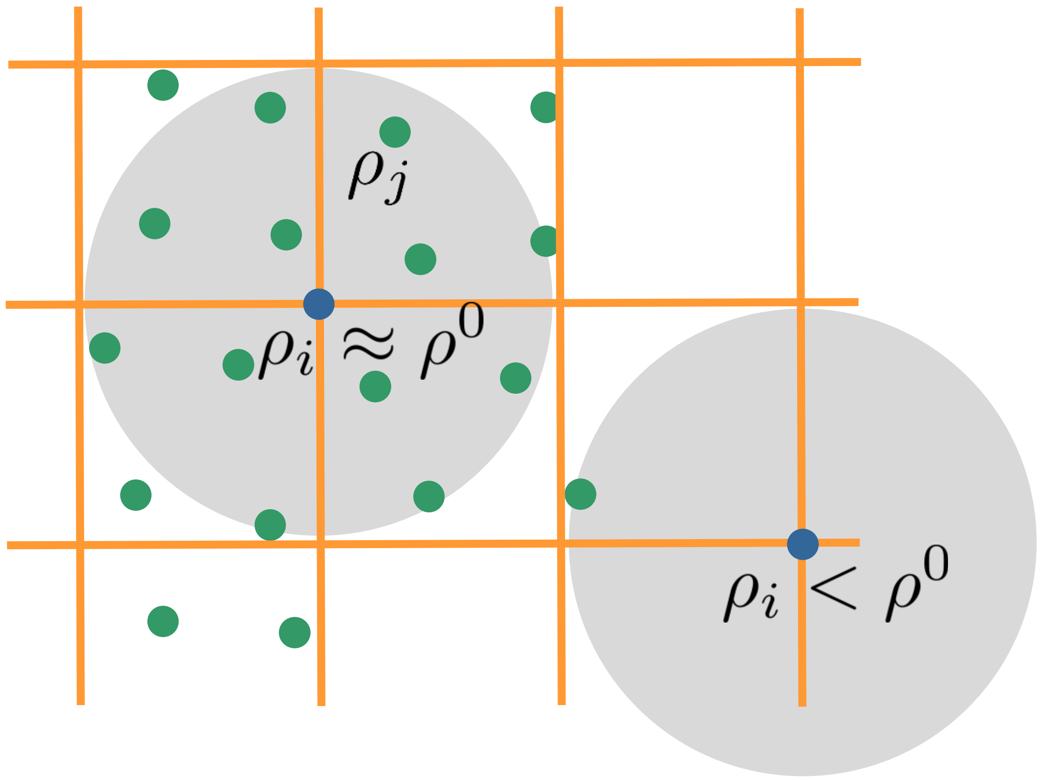
Initialization

– Density computation at grid points using SPH

$$\rho_i = \sum_j \frac{m_j}{\rho_j} \rho_j W_{ij} = \sum_j m_j W_{ij}$$

Grid sample

Particle samples



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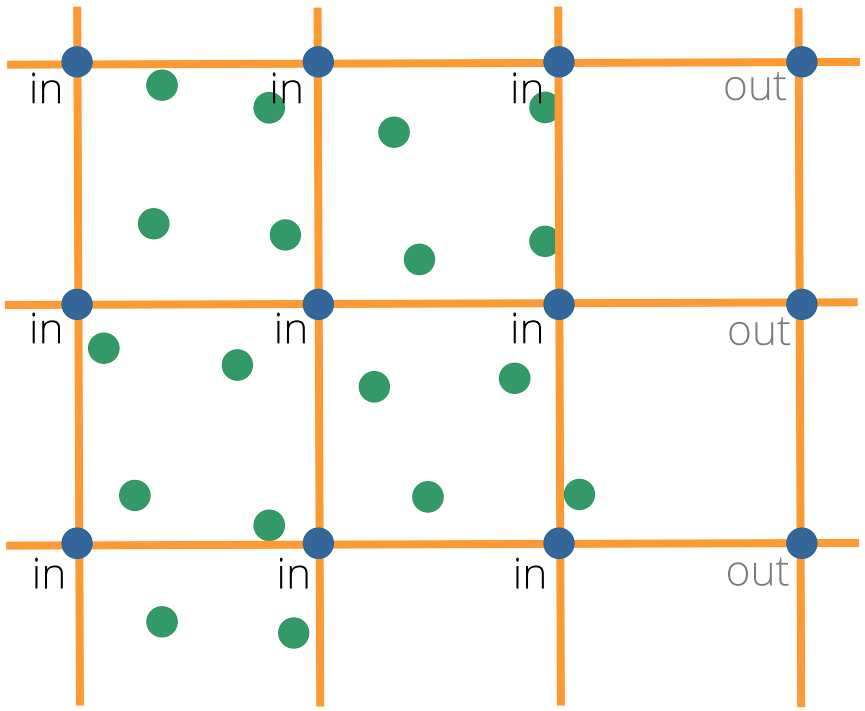
Classification

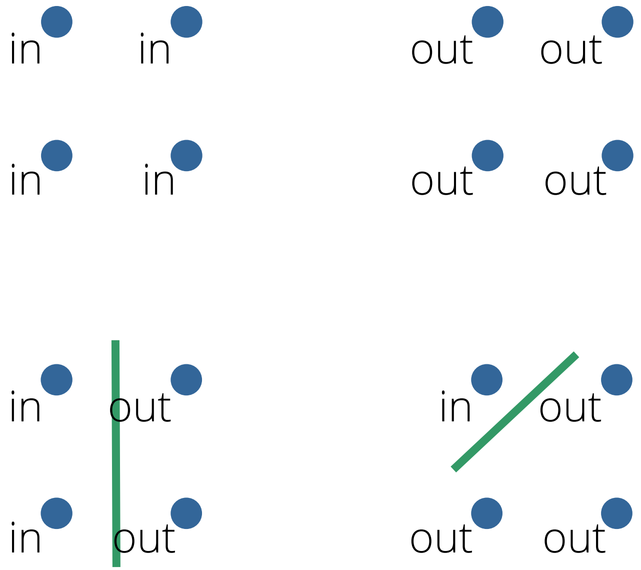
- Inside the fluid: $\rho_i \approx \rho^0$
- Outside: $\rho_i < \rho^0$
- Classification, e.g.

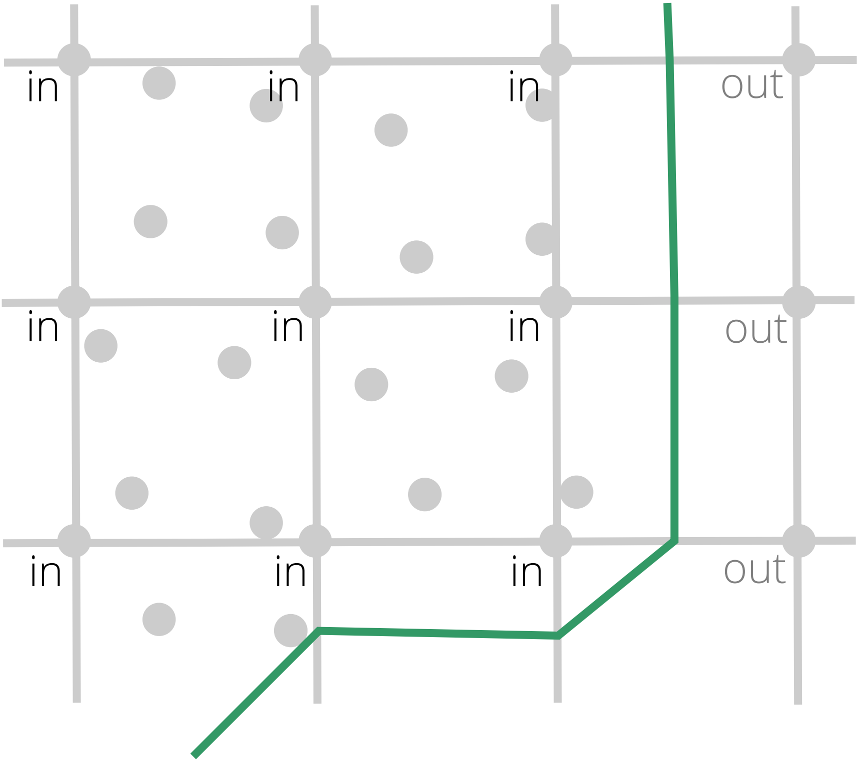
$$\rho_i \leq 0.5\rho^0 \quad \Rightarrow \text{ out}$$
$$\rho_i > 0.5\rho^0 \quad \Rightarrow \text{ in}$$



Iso-Surface Triangulation

– Generate triangles per cell, e.g.





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Outlook

- All materials can be simulated with SPH
  - Fluids
  - Viscous fluids
  - Elastic solids
  - Rigid bodies
- … and their interactions

Gissler et al., ACM Transactions on Graphics, 2019