CONTINUUM MECHANICS

Timo Probst

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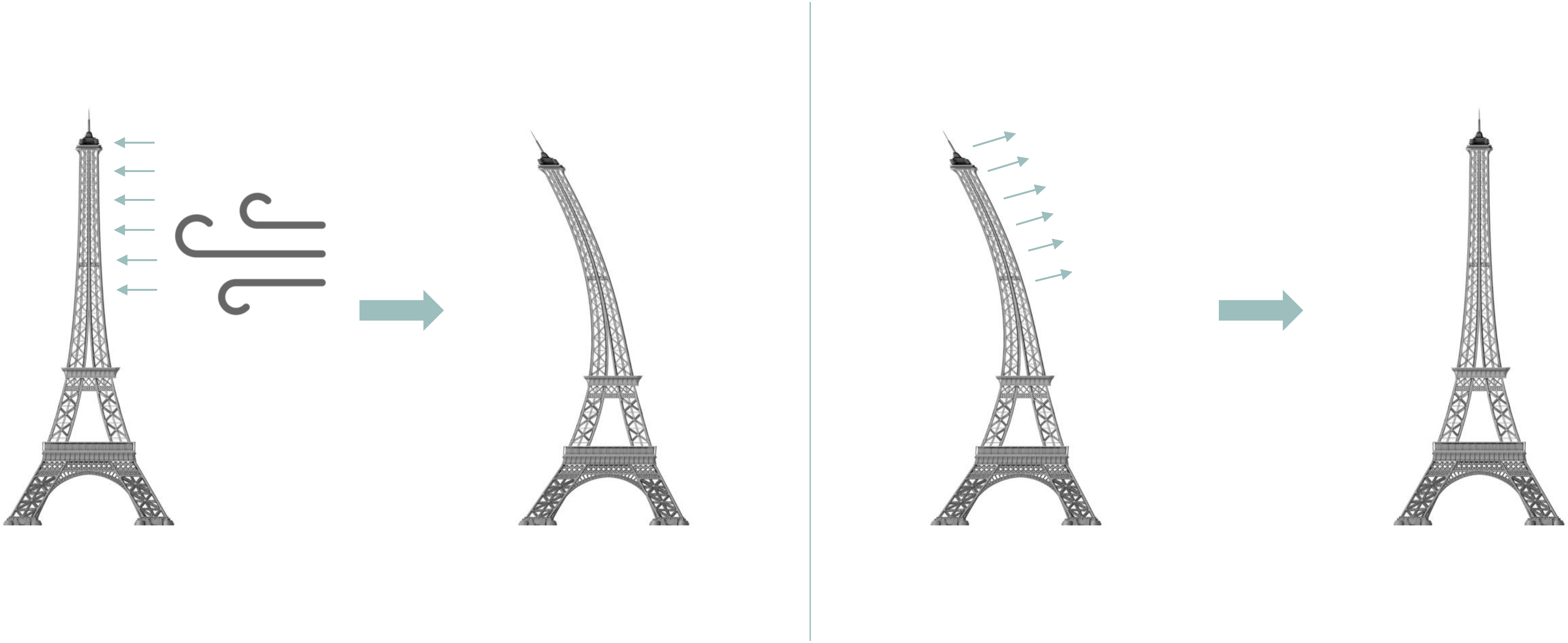


Figure 1: Tower

OUTLINE

- Real-life applications
- Concepts of continuum mechanics
- Elastic materials

REAL-LIFE APPLICATIONS

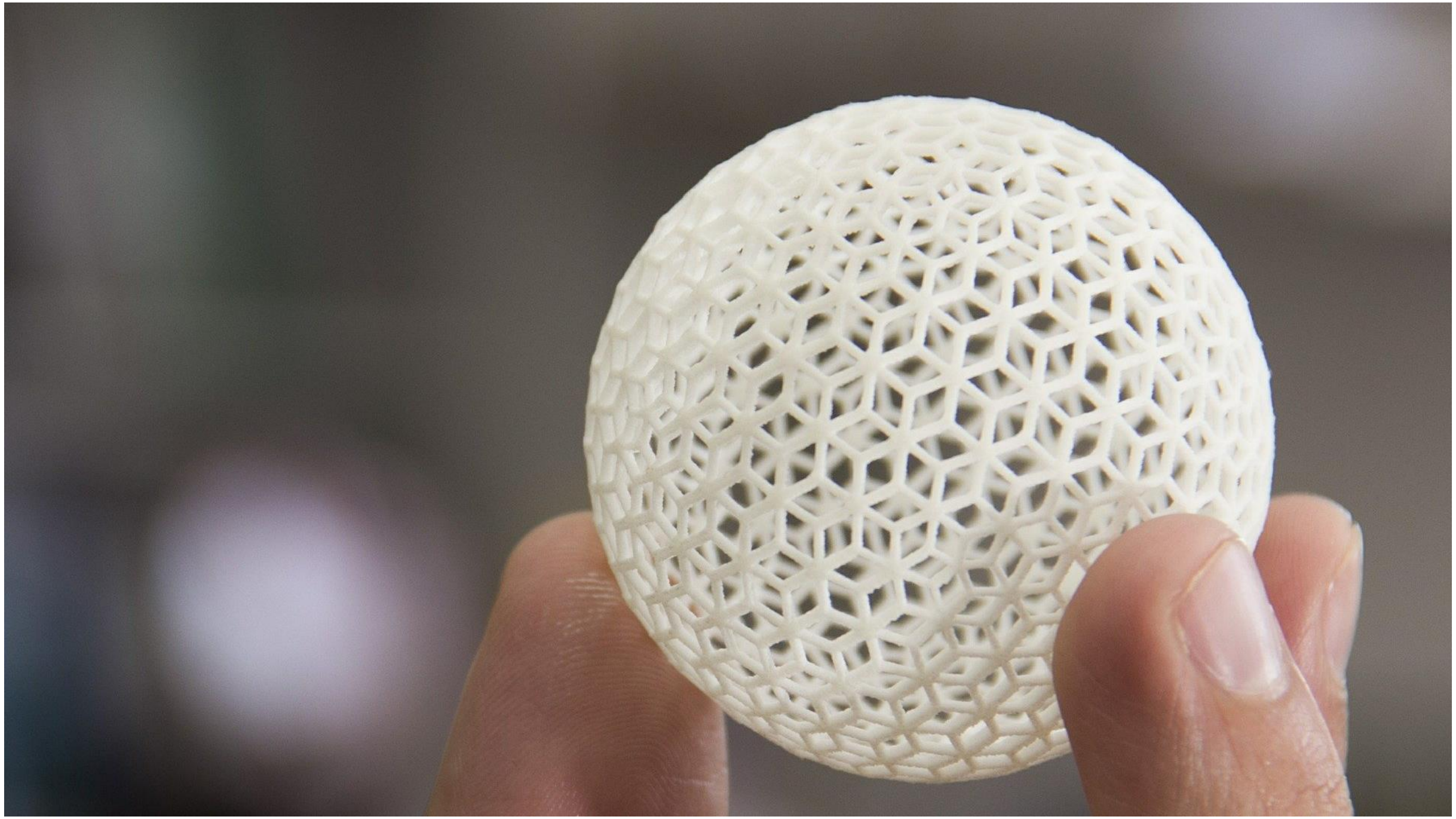


Figure 2: 3D Printing

REAL-LIFE APPLICATIONS

Figure 3: Structural Analysis

REAL-LIFE APPLICATIONS

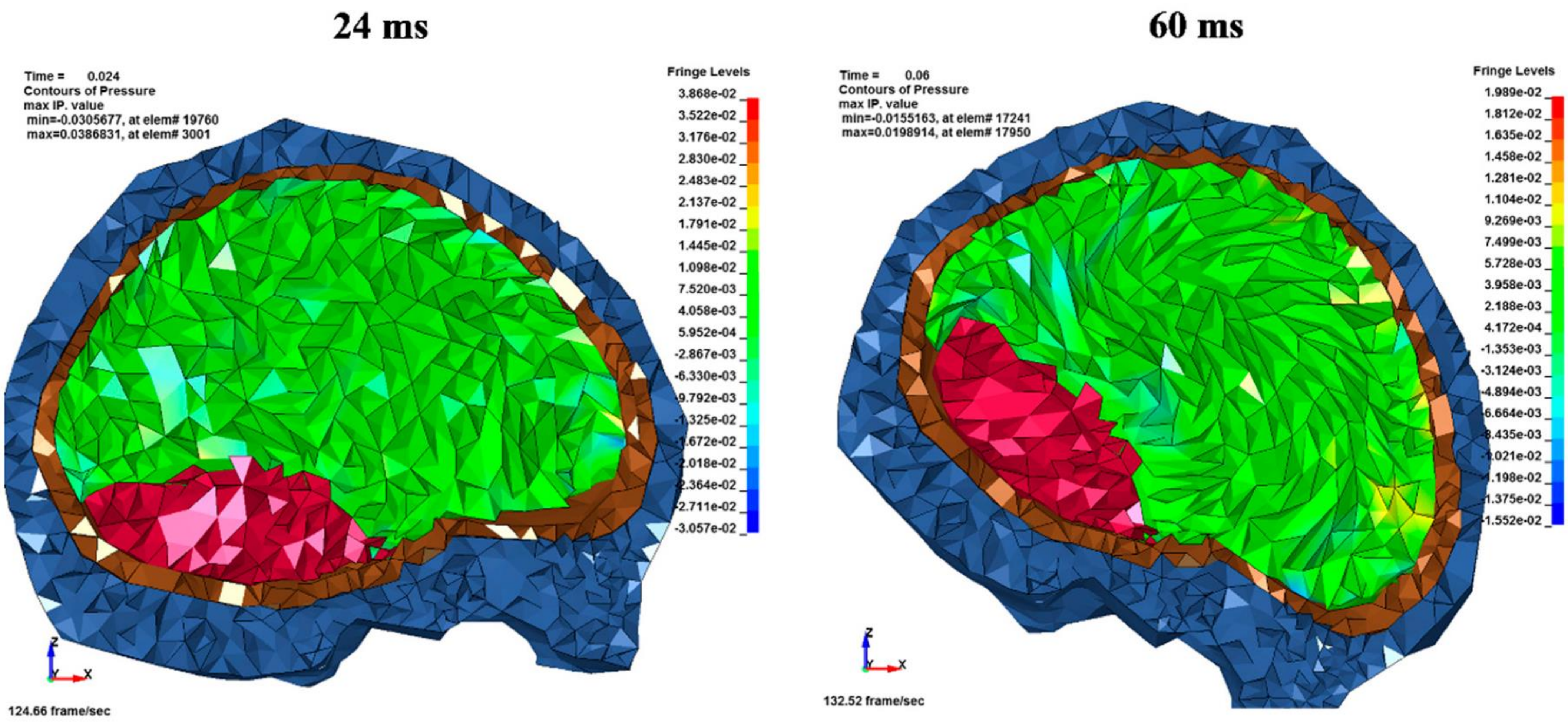


Figure 4: Mechanics of brain tissue

REAL-LIFE APPLICATIONS

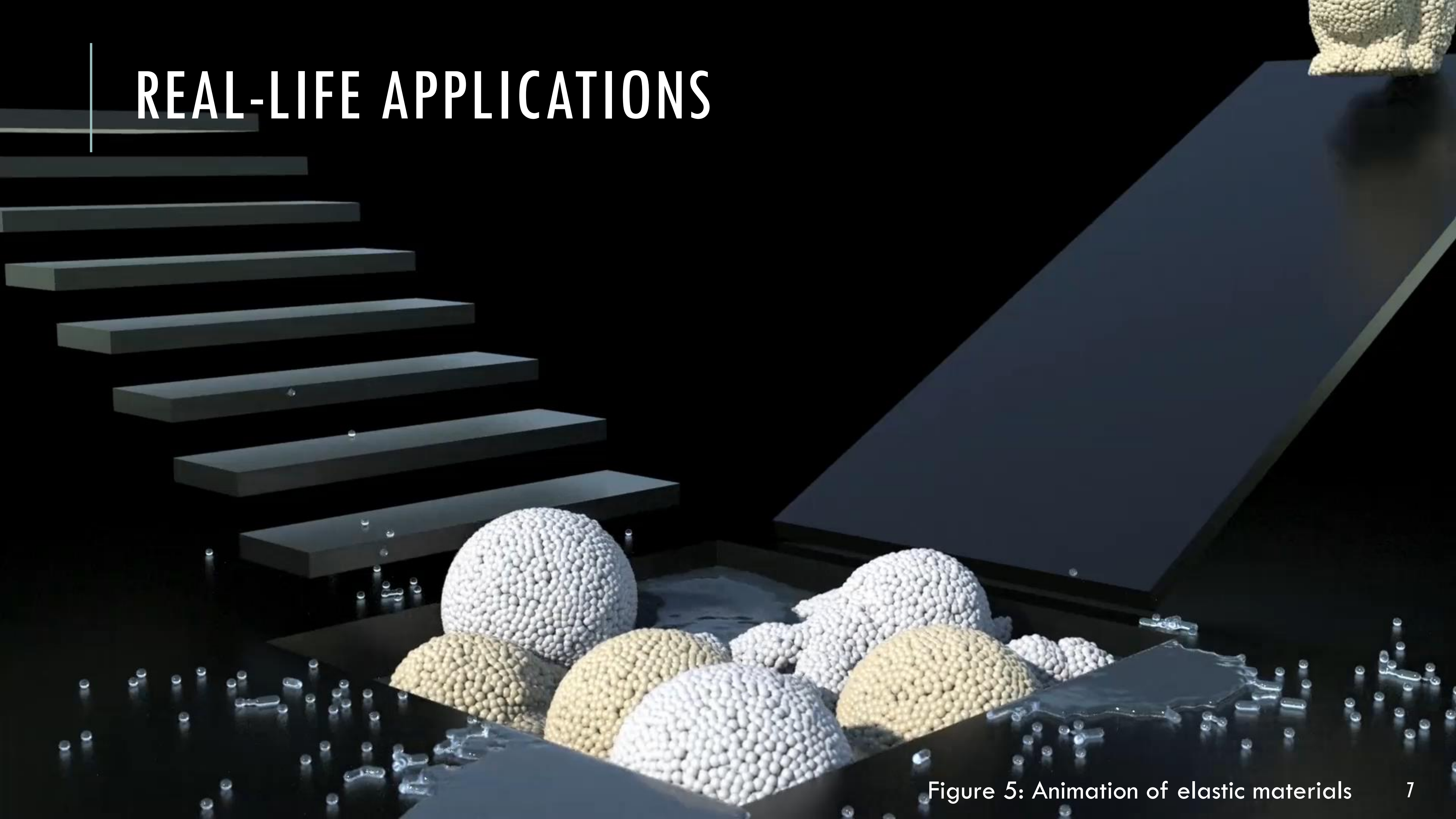


Figure 5: Animation of elastic materials

OUTLINE

- Real-life applications
- Concepts of continuum mechanics
- Elastic materials

CONCEPTS OF CONTINUUM MECHANICS

Studies motion of deformable bodies

- General laws for all materials
- Individual material properties (constitutive equations)
  - Elastic materials
  - Liquids and gases

No molecular structure but continuum

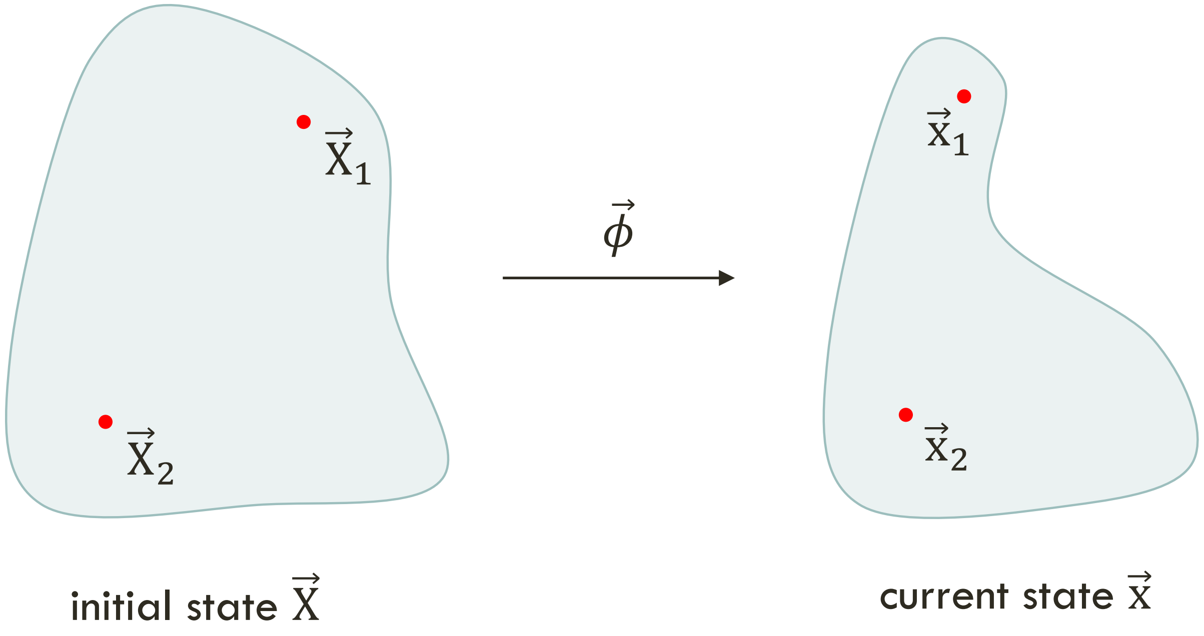
- Density and velocity at each point in space
- Field theory

Our goal: Compute forces at all points in the material depending on the deformation.

DEFORMATION MAP $\vec{\phi}$

Initial positions $\vec{X}$ → current positions $\vec{x}$

$\vec{\phi}(\vec{X}) = \vec{x}$



DEFORMATION GRADIENT $\mathbf{F}$

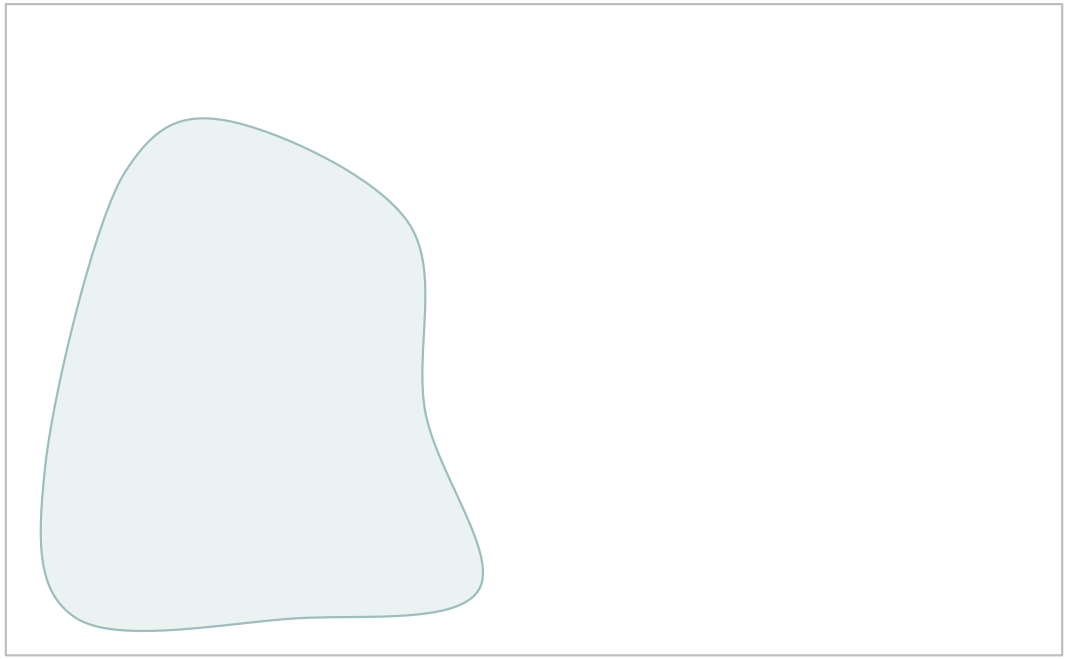
Jacobian of $\vec{\phi}$

$$\mathbf{F}_\phi = \frac{\partial\vec{\phi}}{\partial\vec{\mathrm{X}}} = \begin{pmatrix} \frac{\partial\phi_x}{\partial \mathrm{X}_x} & \frac{\partial\phi_x}{\partial \mathrm{X}_y} & \frac{\partial\phi_x}{\partial \mathrm{X}_z} \\ \frac{\partial\phi_y}{\partial \mathrm{X}_x} & \frac{\partial\phi_y}{\partial \mathrm{X}_y} & \frac{\partial\phi_y}{\partial \mathrm{X}_z} \\ \frac{\partial\phi_z}{\partial \mathrm{X}_x} & \frac{\partial\phi_z}{\partial \mathrm{X}_y} & \frac{\partial\phi_z}{\partial \mathrm{X}_z} \end{pmatrix}$$

EXAMPLES

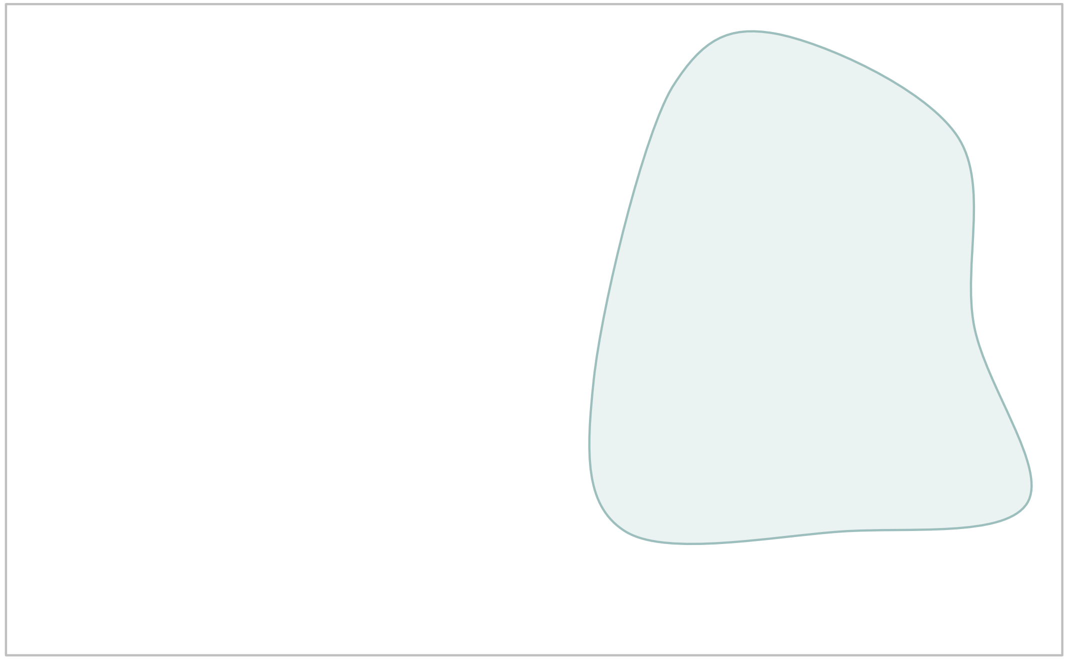
Translation by vector $\vec{t}$

$\vec{\phi}(\vec{X}) = \vec{X} + \vec{t}$ $\mathbf{F} = \mathbf{I}$



initial state $\vec{X}$

$\vec{\phi}$



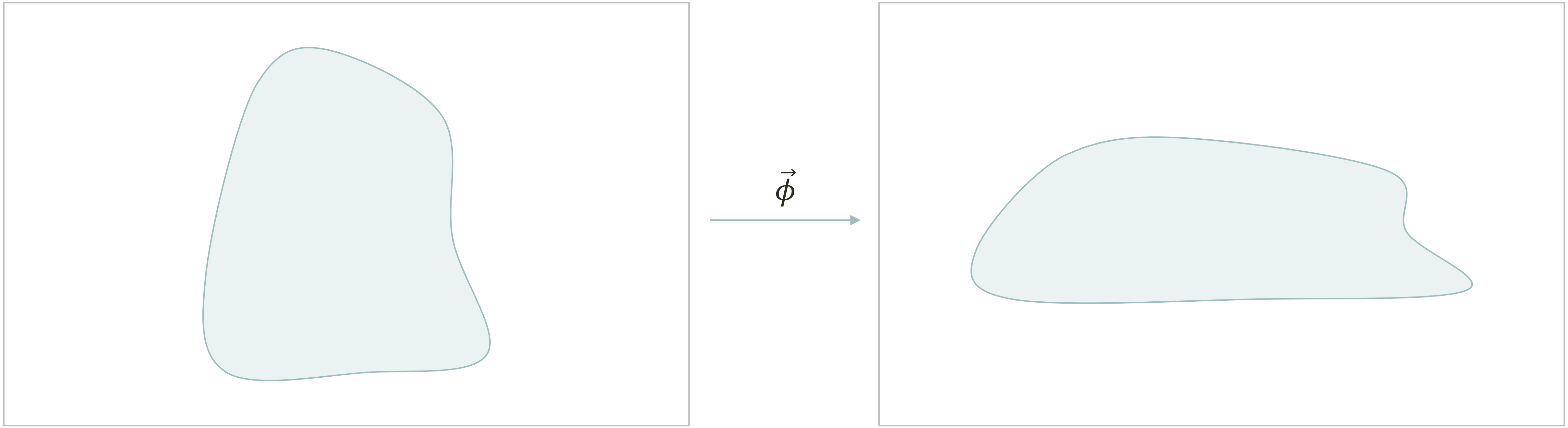
current state $\vec{x}$

EXAMPLES

Non-uniform scaling

$$\vec{\phi}(\vec{\mathrm{X}}) = \begin{pmatrix} 2\mathrm{X}_x \\ 0.5\mathrm{X}_y \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$$



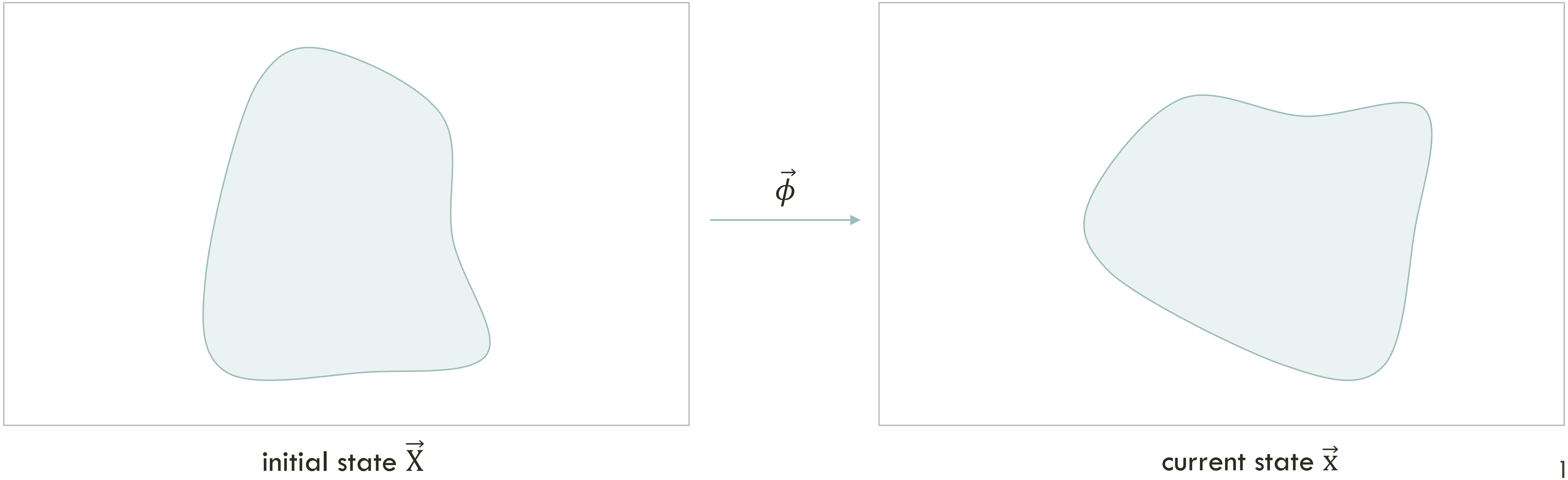
initial state $\vec{\mathrm{X}}$

current state $\vec{\mathrm{x}}$

EXAMPLES

Rotating by a given angle α

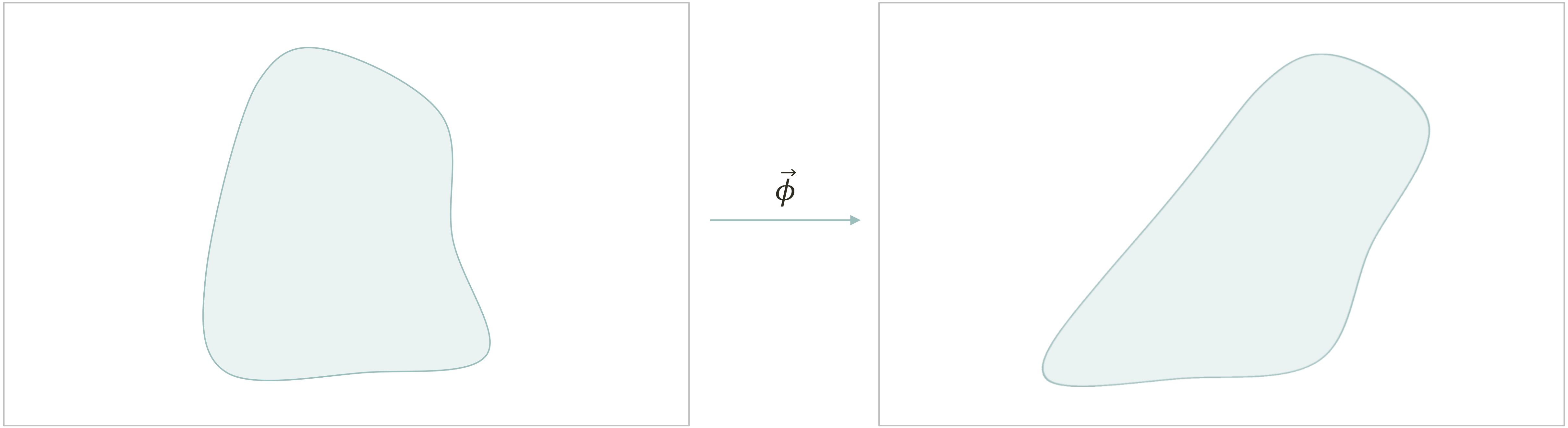
$$\vec{\phi}(\vec{\mathrm{X}}) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathrm{X}_x \\ \mathrm{X}_y \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$



EXAMPLES

Shearing x-coordinate with respect to y-coordinate

$$\vec{\phi}(\vec{\mathrm{X}}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{X}_x \\ \mathrm{X}_y \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



initial state $\vec{\mathrm{X}}$ current state $\vec{\mathrm{x}}$

STRAIN

Description of the deformation

Computed from deformation gradient $\mathbf{F}$

Should exclude rigid body transformations

Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top\mathbf{F} - \mathbf{I})$

Infinitesimal strain tensor $\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{F}^\top + \mathbf{F}) - \mathbf{I}$

- Approximates $\mathbf{E}$ for small deformations (including rotations)
- Linear & faster to compute

STRESS σ

Internal forces that particles of a continuous material exert on each other

Strain-stress relation is given by constitutive equation
$\rightarrow$ material defined

Different materials react differently to strain

Example elastic materials: Hooke’s law for isotropic materials

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Young modulus E and Poisson's ratio v

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{pmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{xz} \\ \epsilon_{xy} \end{pmatrix}$$

Stainless Steel: $E = 1.8 \cdot 10^{11}\ \text{Pa},\ v = 0.3$

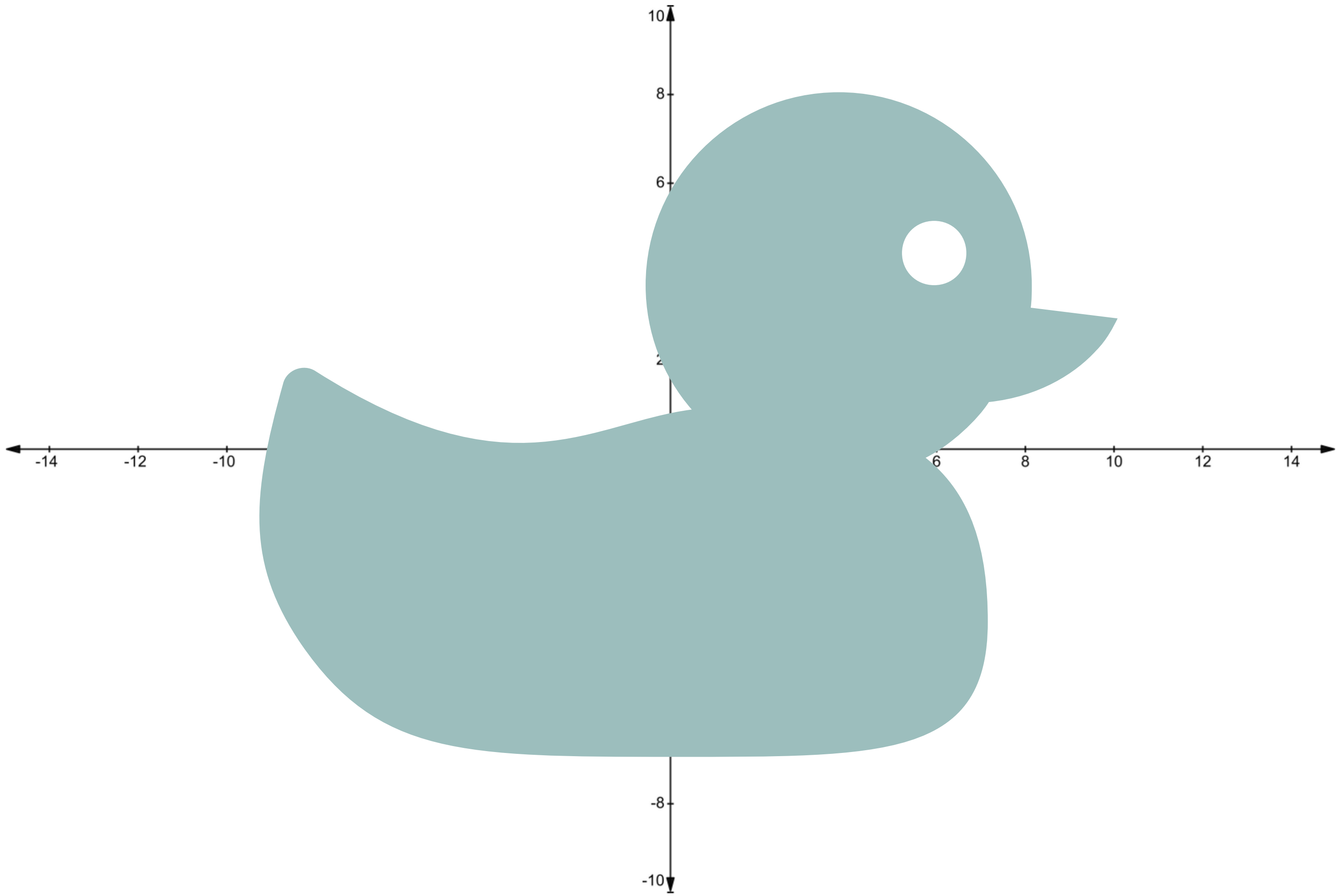
Rubber: $E = 1.0 \cdot 10^{7}\ \text{Pa},\ v = 0.4999$

ACCELERATIONS $\vec{a}$

Cauchy momentum equation

$$\vec{a} = \frac{D\vec{v}}{Dt} = \frac{1}{\rho}\vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{a}_{\text{external}}$$

EXAMPLE: RUBBER DUCK



EXAMPLE: RUBBER DUCK

1. Deformation map

$$\vec{\phi}(\vec{\mathrm{X}}) = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{X}_x \\ \mathrm{X}_y \end{pmatrix}$$

$$= \begin{pmatrix} 0.433 & -0.500 \\ 0.250 & 0.866 \end{pmatrix} \begin{pmatrix} \mathrm{X}_x \\ \mathrm{X}_y \end{pmatrix}$$

2. Deformation Gradient

$$\mathbf{F} = \frac{\partial \vec{\phi}}{\partial \vec{\mathrm{X}}} = \begin{pmatrix} 0.433 & -0.500 \\ 0.250 & 0.866 \end{pmatrix}$$

3. Strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}) = \begin{pmatrix} 0.250 & 0 \\ 0 & 1 \end{pmatrix}$$

EXAMPLE: RUBBER DUCK

3. Strain

$$\mathbf{E} = \begin{pmatrix} 0.250 & 0 \\ 0 & 1 \end{pmatrix}$$

4. Stress

Rubber: $E = 1.0 \cdot 10^7$ Pa, $v = 0.4999$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix} \begin{pmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{xy} \end{pmatrix} = \begin{pmatrix} 2.08 \cdot 10^{10} \\ 2.08 \cdot 10^{10} \\ 0 \end{pmatrix} \mathrm{Pa}$$

$$\boldsymbol{\sigma} = \begin{pmatrix} 2.08 \cdot 10^{10} & 0 \\ 0 & 2.08 \cdot 10^{10} \end{pmatrix} \mathrm{Pa}$$

EXAMPLE: RUBBER DUCK

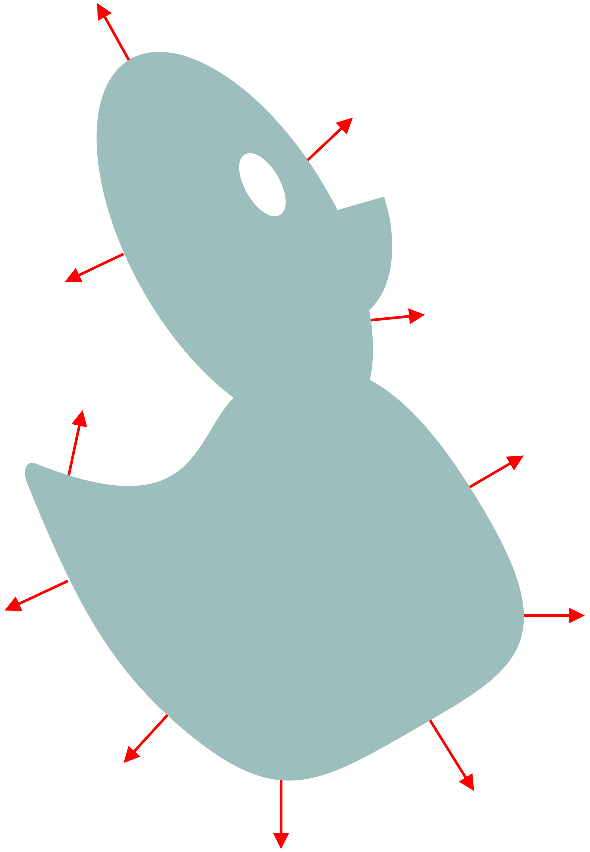
4. Stress

$$\boldsymbol{\sigma} = \begin{pmatrix} 2.08 \cdot 10^{10} & 0 \\ 0 & 2.08 \cdot 10^{10} \end{pmatrix}$$

5. Accelerations in the material

$$\vec{a} = \frac{1}{\rho}\vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{a}_{external} = \frac{1}{\rho}\vec{\nabla} \cdot \begin{pmatrix} 2.08 \cdot 10^{10} & 0 \\ 0 & 2.08 \cdot 10^{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Accelerations on the surface of the material
→ non-zero divergence of the stress
→ non-zero accelerations



CONCEPTS - SUMMARY

Deformation map $\vec{\phi}$

Gradient

Deformation gradient $\mathbf{F}$

Definition, e.g. Green strain tensor or infinitesimal strain tensor

Strain $\mathbf{E}$, $\boldsymbol{\epsilon}$

Constitutive equation (material defined)

Stress $\boldsymbol{\sigma}$

Cauchy momentum equation

Acceleration $\vec{\mathrm{a}}$

OUTLINE

- Real-life applications
- Concepts of continuum mechanics
- Elastic materials

ELASTIC MATERIALS

Finite elements as a discretization method

Material is subdivided into tetrahedrons

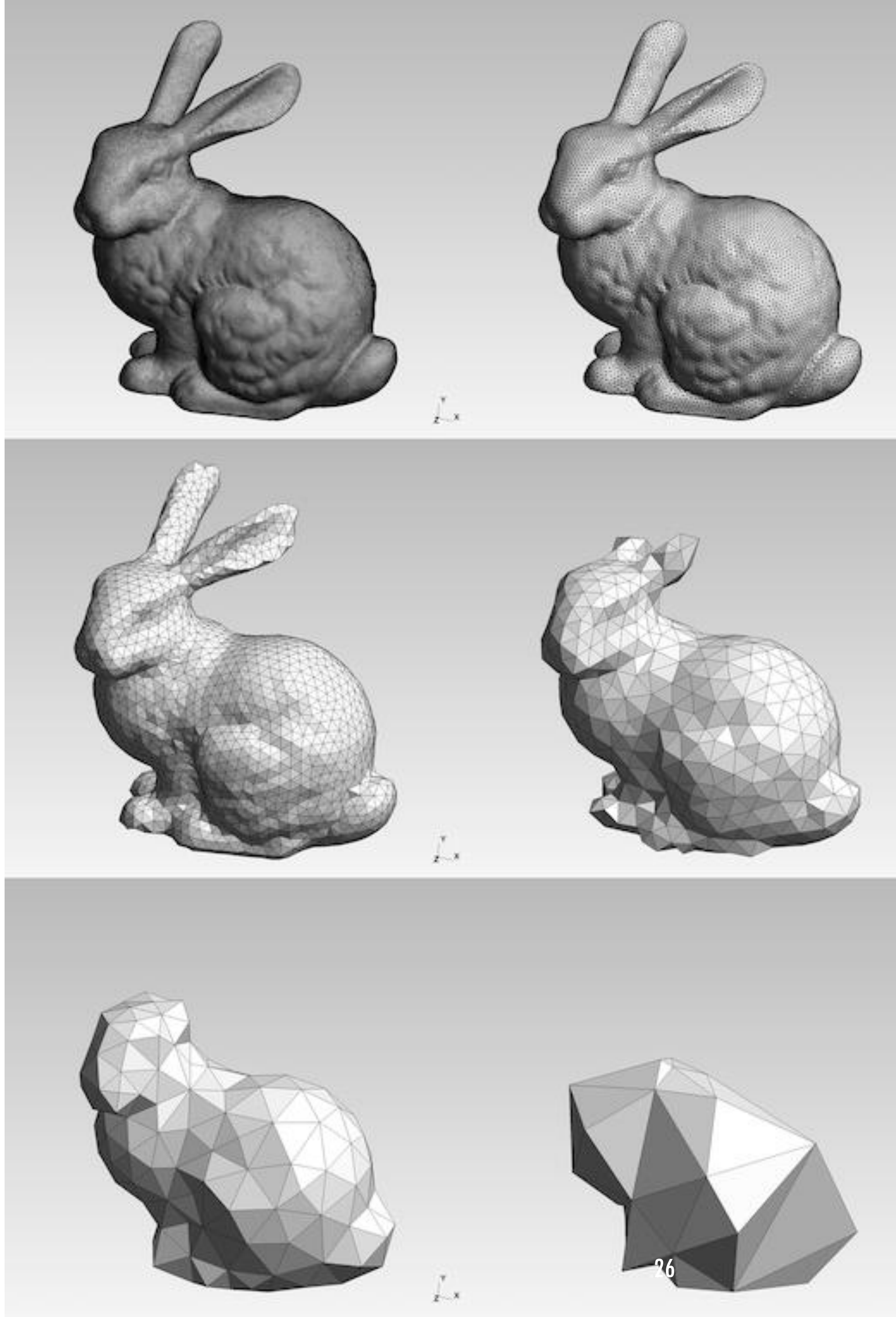
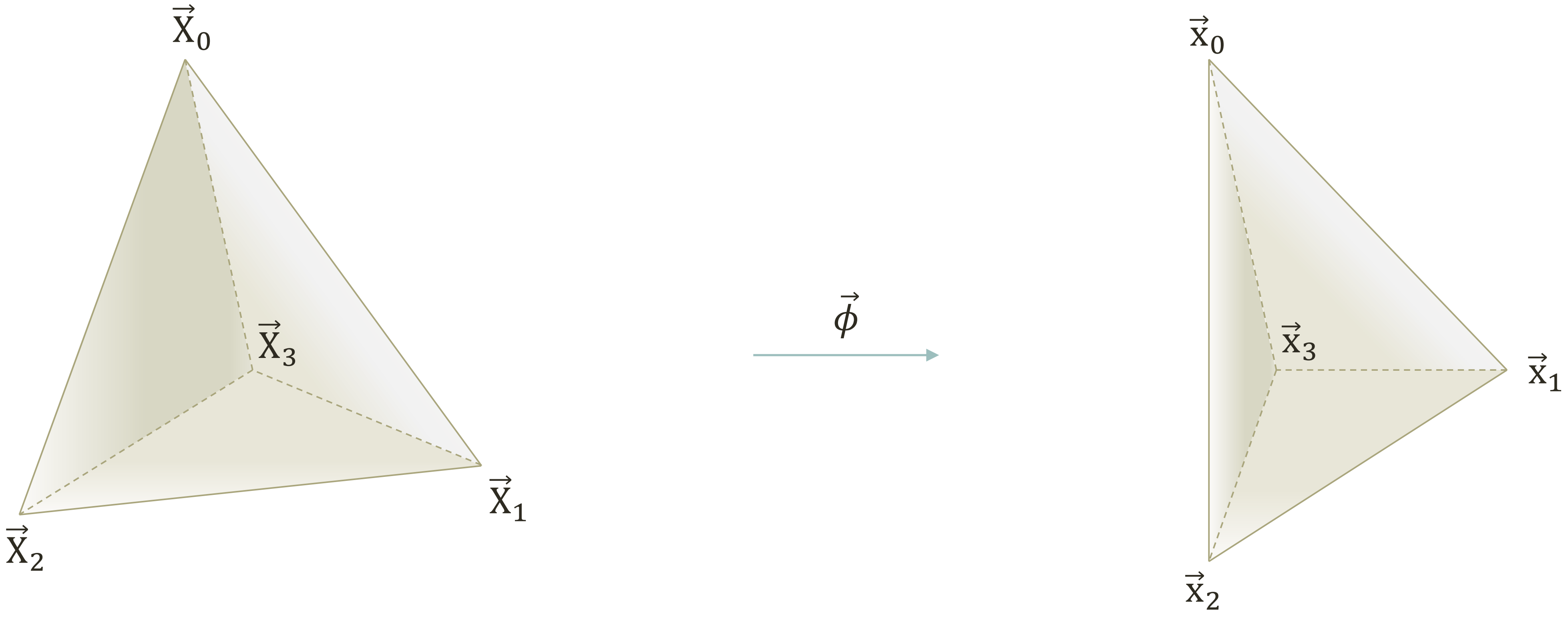


Figure 6: Finite element mesh

ELASTIC MATERIALS

Goal: Compute forces at all vertices depending on deformation



ELASTIC MATERIALS

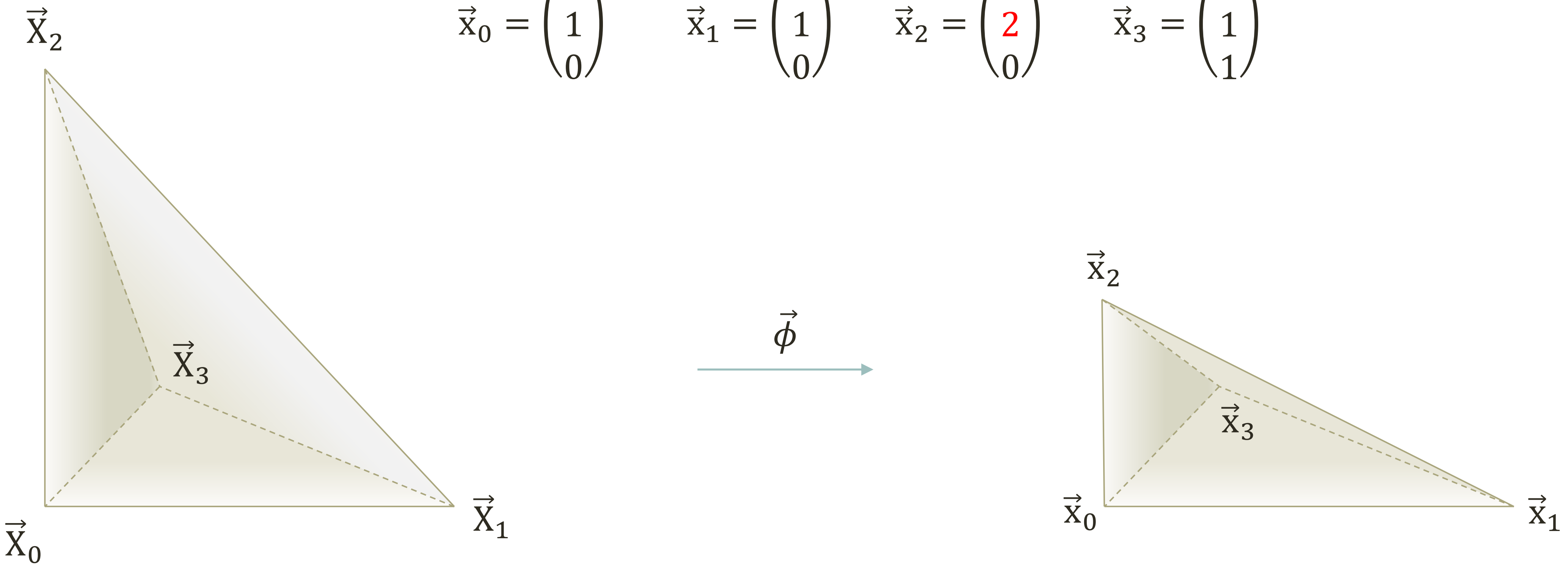
Simulation step:

1. Translate $\vec{\mathrm{X}}$ and $\vec{\mathrm{x}}$ such that $\vec{\mathrm{X}}_0 = \vec{\mathrm{x}}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
2. Compute deformation map $\vec{\phi}(\vec{\mathrm{X}}) = [\vec{\mathrm{x}}_1, \vec{\mathrm{x}}_2, \vec{\mathrm{x}}_3][\vec{\mathrm{X}}_1, \vec{\mathrm{X}}_2, \vec{\mathrm{X}}_3]^{-1}\vec{\mathrm{X}}$
3. Compute deformation gradient $\mathbf{F} = [\vec{\mathrm{x}}_1, \vec{\mathrm{x}}_2, \vec{\mathrm{x}}_3][\vec{\mathrm{X}}_1, \vec{\mathrm{X}}_2, \vec{\mathrm{X}}_3]^{-1}$
4. Compute strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\top}\mathbf{F} - \mathbf{I})$
5. Compute stress $\boldsymbol{\sigma}(\mathbf{E})$ (Hooke's law)
6. Compute forces at surface $\vec{\mathrm{n}}$ $\vec{f}(\vec{\mathrm{n}}) = A \cdot \boldsymbol{\sigma} \cdot \vec{\mathrm{n}}$
7. Equally distribute surface forces over $\vec{\mathrm{x}}_i$

EXAMPLE: TETRAHEDRON

$$\vec{X}_0 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \vec{X}_1 = \begin{pmatrix}3\\1\\0\end{pmatrix} \quad \vec{X}_2 = \begin{pmatrix}1\\3\\0\end{pmatrix} \quad \vec{X}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\vec{x}_0 = \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \vec{x}_1 = \begin{pmatrix}3\\1\\0\end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix}1\\2\\0\end{pmatrix} \quad \vec{x}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$



EXAMPLE: TETRAHEDRON

1. Translate to origin

$$\vec{X}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{X}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{X}_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \qquad \vec{X}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{x}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2. Compute deformation map $\vec{\phi}$

$$\vec{\phi}(\vec{X}) = [\vec{x}_1, \vec{x}_2, \vec{x}_3][\vec{X}_1, \vec{X}_2, \vec{X}_3]^{-1}\vec{X} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\vec{X}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}\vec{X}$$

EXAMPLE: TETRAHEDRON

3. Compute deformation gradient $\mathbf{F}$

$$\mathbf{F} = \frac{\partial \vec{\phi}}{\partial \vec{\mathrm{X}}} = \frac{\partial}{\partial \vec{\mathrm{X}}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\mathrm{X}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Compute strain $\mathbf{E}$

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbf{I}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.375 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

5. Compute stress $\boldsymbol{\sigma}$ using Hooke's law with $E = 1.0 \cdot 10^7$ Pa, $v = 0.4999$

$$\boldsymbol{\sigma} = \begin{pmatrix} -6.249 \cdot 10^9 & 0 & 0 \\ 0 & -6.251 \cdot 10^9 & 0 \\ 0 & 0 & -6.249 \cdot 10^9 \end{pmatrix} \mathrm{Pa}$$

EXAMPLE: TETRAHEDRON

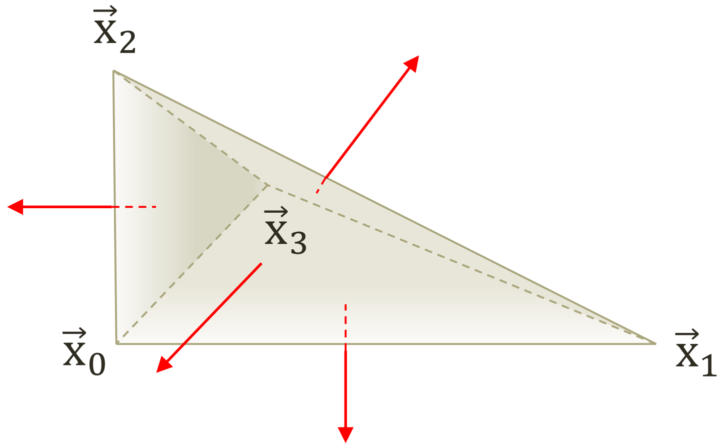
5. Compute forces $\vec{f}$ at all surfaces with $\vec{f}(\vec{\mathrm{n}}) = A \cdot \boldsymbol{\sigma} \cdot \vec{\mathrm{n}}$

$$\vec{f}_{012} = 1\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6.249 \cdot 10^9 \end{pmatrix} \mathrm{N}$$

$$\vec{f}_{013} = 1\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6.251 \cdot 10^9 \\ 0 \end{pmatrix} \mathrm{N}$$

$$\vec{f}_{023} = 0.5\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3.125 \cdot 10^9 \\ 0 \\ 0 \end{pmatrix} \mathrm{N}$$

$$\vec{f}_{123} = \frac{3}{2}\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} 3.125 \cdot 10^9 \\ 6.251 \cdot 10^9 \\ 6.249 \cdot 10^9 \end{pmatrix} \mathrm{N}$$



EXAMPLE: TETRAHEDRON

5. Compute forces $\vec{f}$ at all surfaces with $\vec{f}(\vec{\mathrm{n}}) = A \cdot \boldsymbol{\sigma} \cdot \vec{\mathrm{n}}$

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$$\vec{f}_{013} = 1\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6.251 \cdot 10^9 \\ 0 \end{pmatrix} \mathrm{N}$$

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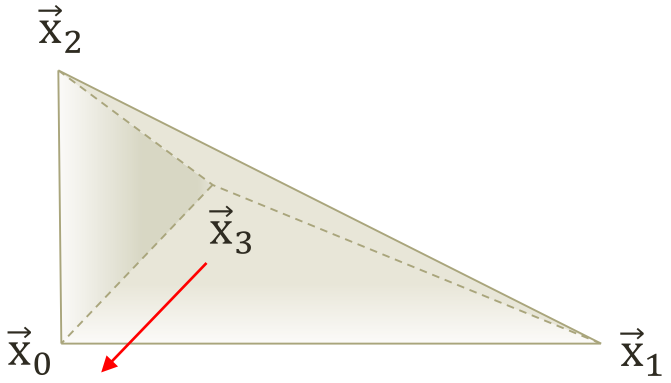

EXAMPLE: TETRAHEDRON

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$$\vec{f}_{013} = 1\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6.251 \cdot 10^9 \\ 0 \end{pmatrix} \mathrm{N}$$

$$\vec{f}_{023} = 0.5\mathrm{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3.125 \cdot 10^9 \\ 0 \\ 0 \end{pmatrix} \mathrm{N}$$

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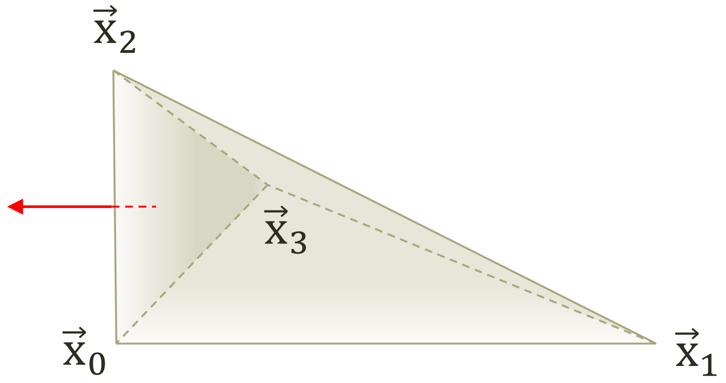


EXAMPLE: TETRAHEDRON

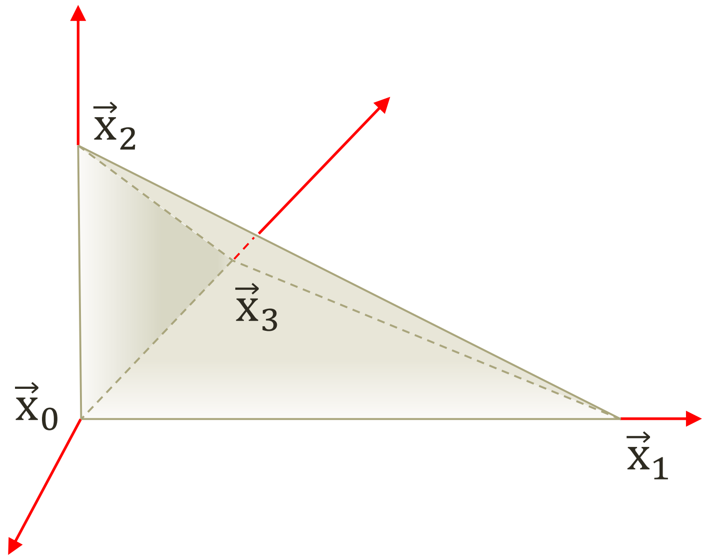
6. Equally distribute surface forces over vertices

$$\vec{f}(\vec{x}_0) = \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{013} + \vec{f}_{023}\right) = \begin{pmatrix} -1.04 \\ -2.08 \\ -2.08 \end{pmatrix} \cdot 10^9 \text{ N}$$

$$\vec{f}(\vec{x}_1) = \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{013} + \vec{f}_{123}\right) = \begin{pmatrix} 1.04 \\ 0 \\ 0 \end{pmatrix} \cdot 10^9 \text{ N}$$

$$\vec{f}(\vec{x}_2) = \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 2.08 \\ 0 \end{pmatrix} \cdot 10^9 \text{ N}$$

$$\vec{f}(\vec{x}_3) = \frac{1}{3}\left(\vec{f}_{013} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 0 \\ 2.08 \end{pmatrix} \cdot 10^9 \text{ N}$$



EXAMPLE: TETRAHEDRON

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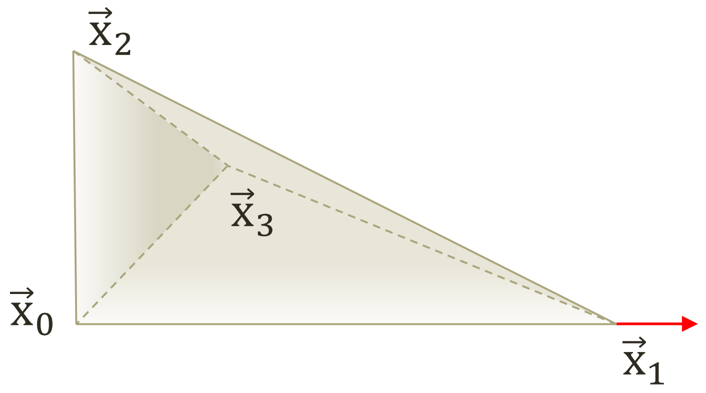

EXAMPLE: TETRAHEDRON

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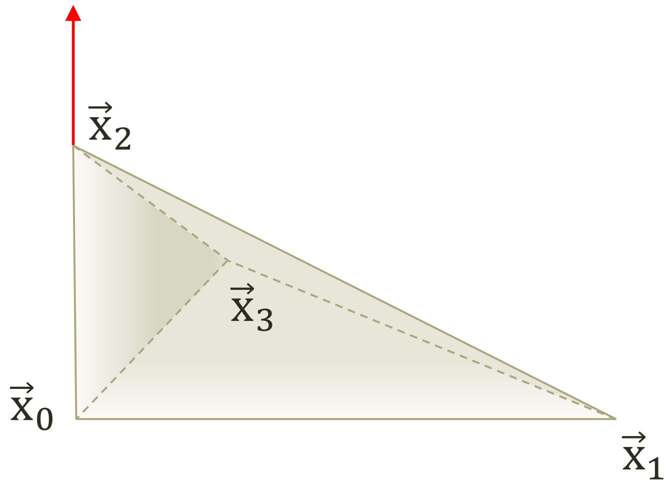
$$\vec{f}(\vec{x}_3) = \frac{1}{3}\left(\vec{f}_{013} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 0 \\ 2.08 \end{pmatrix} \cdot 10^9 \text{ N}$$



EXAMPLE: TETRAHEDRON

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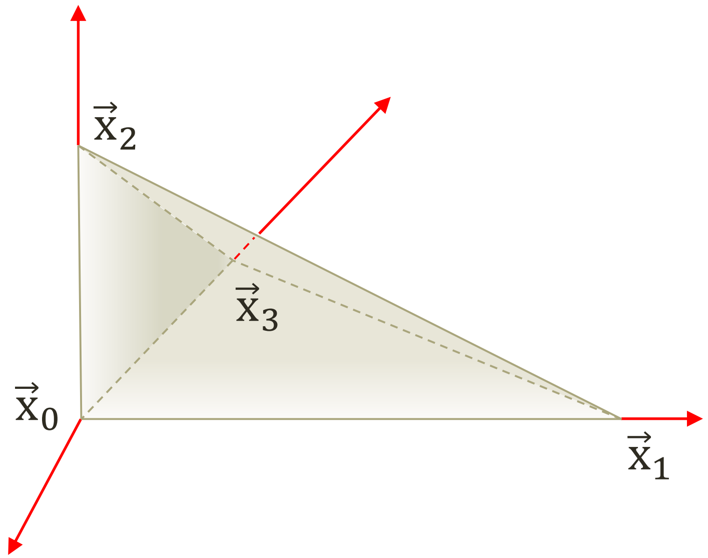
$$\vec{f}(\vec{x}_0) = \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{013} + \vec{f}_{023}\right) = \begin{pmatrix} -1.04 \\ -2.08 \\ -2.08 \end{pmatrix} \cdot 10^9 \text{ N}$$

$$\vec{f}(\vec{x}_1) = \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{013} + \vec{f}_{123}\right) = \begin{pmatrix} 1.04 \\ 0 \\ 0 \end{pmatrix} \cdot 10^9 \text{ N}$$

$$\vec{f}(\vec{x}_2) = \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 2.08 \\ 0 \end{pmatrix} \cdot 10^9 \text{ N}$$

$$\vec{f}(\vec{x}_3) = \frac{1}{3}\left(\vec{f}_{013} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 0 \\ 2.08 \end{pmatrix} \cdot 10^9 \text{ N}$$



SUMMARY

Continuum Mechanics have many real-life applications.

Can be used to simulate a wide range of materials

- Similar procedure for all materials possible

For implementation we need discretization methods

- Finite Elements
- SPH
- MPM

REFERENCES

Peer, A., Gissler, C., Band, S., & Teschner, M. (2018, September). An implicit SPH formulation for incompressible linearly elastic solids. In *Computer Graphics Forum* (Vol. 37, No. 6, pp. 135-148).

Ratajczak, M., Ptak, M., Chybowski, L., Gawdzińska, K., & Będziński, R. (2019). Material and structural modeling aspects of brain tissue deformation under dynamic loads. *Materials*, *12*(2), 271.

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RESOURCES

Figure 1: Tower, GDJ, Web, accessed 5.6.2020 in https://openclipart.org/detail/233894/detailed-eiffel-tower-trace-2

Figure 2: 3D Printing, metalurgiamontemar0, Web, accessed 05.06.2020 in https://pixabay.com/de/photos/ball-3d-druck-gestaltung-597523/

Figure 3: Structural Analysis, Web, accessed 05.06.2020 in https://pixabay.com/de/photos/golden-gate-br%C3%BCcke-san-francisco-388917/

Figure 4: Mechanics of brain tissue from Ratajczak, M., Ptak, M., Chybowski, L., Gawdzińska, K., & Będziński, R. (2019). Material and structural modeling aspects of brain tissue deformation under dynamic loads. *Materials*, *12*(2), 271.

Figure 5: Animation of elastic materials from Peer, A., Gissler, C., Band, S., & Teschner, M. (2018, September). An implicit SPH formulation for incompressible linearly elastic solids. In *Computer Graphics Forum* (Vol. 37, No. 6, pp. 135-148).

Figure 6: Finite element mesh, Web, accessed 14.06.2020 in https://gmsh.info/

FLUIDS

Navier-Stokes equation for incompressible fluids

$$\vec{\mathrm{a}} = -\frac{1}{\rho}\vec{\nabla}p + \frac{\eta}{\rho}\vec{\nabla}^2\vec{\mathrm{v}} + \vec{\mathrm{a}}_{\mathrm{external}}$$

Cauchy momentum equation

$$\vec{\mathrm{a}} = \frac{1}{\rho}\vec{\nabla}\cdot\boldsymbol{\sigma} + \vec{\mathrm{a}}_{\mathrm{external}}$$

Stress in fluids:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \eta\left(\vec{\nabla}\vec{\mathrm{v}} + \left(\vec{\nabla}\vec{\mathrm{v}}\right)^{\mathsf{T}}\right)$$

PRESSURE

Strain $\mathbf{E}$ corresponds to density ρ

$$\mathbf{E} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

Pressure stress $\boldsymbol{\sigma}$ corresponds to negative pressure p

$$\boldsymbol{\sigma} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

Constitutive equation to relate strain and stress is a state equation:

$$p = k \cdot \rho$$

$$p = k_1 \cdot \left(\frac{\rho}{\rho_0} - 1\right)^{k_2}$$

VISCOSITY

Deformation can also depend on a velocity field that doesn't match a rest velocity field.

Strain rate tensor $\mathbf{E}$

$$\mathbf{E} = \frac{1}{2}\left(\vec{\nabla}\vec{\mathrm{v}} + \vec{\nabla}\vec{\mathrm{v}}^{\top}\right)$$

In Newtonian fluids strain and stress are linearly dependent with 2η.

Viscous stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = 2\eta \cdot \mathbf{E} = \eta\left(\vec{\nabla}\vec{\mathrm{v}} + \vec{\nabla}\vec{\mathrm{v}}^{\top}\right)$$