Albert-Ludwigs-University Freiburg
Faculty of Engineering
Department of Computer Science
Chair for Computer Graphics

Seminar Report

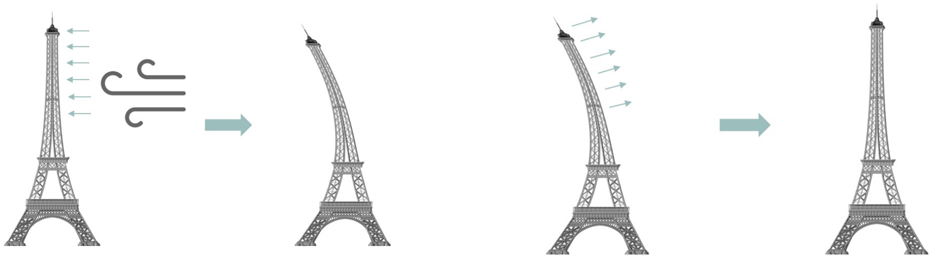
Continuum Mechanics

Timo Probst

July 12, 2020

Supervised by Prof. Dr.-Ing. Matthias Teschner

Contents



Computing deformations depending on external forces

Computing forces in the material depending on the deformation of the object

Figure 1: Two scenarios that can be solved using continuum mechanics

1 Introduction

Continuum mechanics is a subfield of mechanics that studies the motion of deformable bodies. One can think of two scenarios where continuum mechanics are used, illustrated in figure 1: On the one hand we can use continuum mechanics to simulate the deformation of deformable materials if some external forces are applied to the object. On the other hand, we might also be interested in the forces that act at all points in the material due to some deformation of the deformable object.

In real-life there are many use cases for continuum mechanics. In 3D printing, one can analyze the strength of the 3D structure of a model and thus find weak points in the model that need further improvement before printing the object. The same principle holds true for much bigger objects like architecture. Here, it is also important to simulate deformations on bridges and towers when subjected to forces resulting from gravity and wind, in order to guarantee a robust building. There are also medical applications, as an example Ratajczak et al. simulated the effects of impacts against the head on the brain using continuum mechanics [RPC+19]. They were able to compute deformations of the brain tissue and thus predict where the tissue will be injured. This principle can also be applied in order to simulate the strength of bones as shown by George et al. in [GAR18]. Another interesting application area of continuum mechanics is the animation of all kind of deformable materials. As an example, Peer et al. showed a method to simulate elastic materials using continuum mechanics that also works well together with other materials such as liquids [PGBT18].

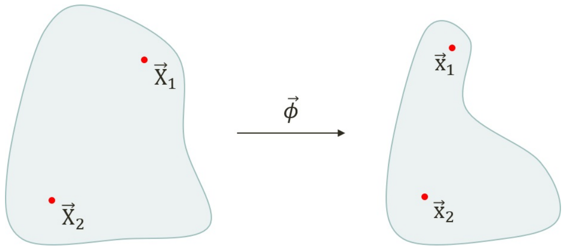


Figure 2: Some deformation is applied to the material which is expressed by the deformation map $\vec{\phi}$. Two exemplary initial configuration points $\vec{X}_1$ and $\vec{X}_2$ and their current positions $\vec{x}_1$ and $\vec{x}_2$ are shown.

2 Concepts

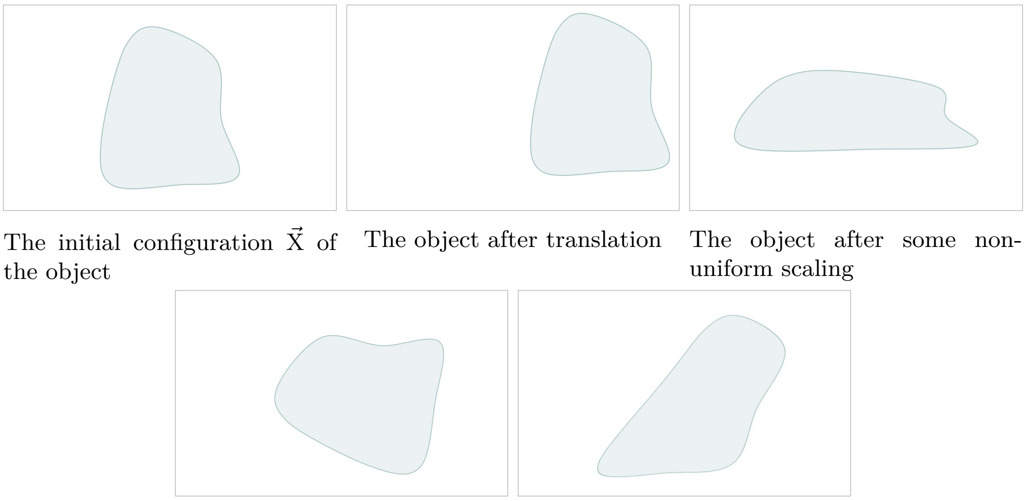
As already mentioned, in continuum mechanics we study the motion of all kind of deformable materials. This will include general laws and principles that are the same for all materials and some material individual properties that define the individual behavior of materials. One important thing to keep in mind is that in continuum mechanics we don't model the underlying molecular structure of a material but instead assume that the material is a continuum. This means there is a density and velocity at each point in space what makes continuum mechanics a field theory. The goal of this section is to show how to compute forces in a deformable body, depending on a deformation of this body.

2.1 Deformation Map & Deformation Gradient

We will start by defining the deformation in the material with a *deformation map* $\vec{\phi}(\vec{X})$ that maps all points in the material from an initial rest configuration $\vec{X}$ to a current configuration of the material $\vec{x}$. This is also illustrated in figure 2. From the deformation map $\vec{\phi}$ we can extract the *deformation gradient* $\mathbf{F}$ which is simply the Jacobian of $\vec{\phi}$, a matrix that consists of partial derivatives of all directions of $\vec{\phi}$ with respect to all coordinate directions. In the case of three dimensions $\mathbf{F}$ then looks like

$$\mathbf{F}(\vec{\phi}(\vec{X})) = \begin{pmatrix} \frac{\partial\phi(\vec{X})_x}{\partial x} & \frac{\partial\phi(\vec{X})_x}{\partial y} & \frac{\partial\phi(\vec{X})_x}{\partial z} \\ \frac{\partial\phi(\vec{X})_y}{\partial x} & \frac{\partial\phi(\vec{X})_y}{\partial y} & \frac{\partial\phi(\vec{X})_y}{\partial z} \\ \frac{\partial\phi(\vec{X})_z}{\partial x} & \frac{\partial\phi(\vec{X})_z}{\partial y} & \frac{\partial\phi(\vec{X})_z}{\partial z} \end{pmatrix}. \tag{2.1}$$

To improve the intuition behind deformation map and deformation gradient, we will look at four examples of transformations represented in $\vec{\phi}(\vec{X})$ and their corresponding deformation gradients $\mathbf{F}$. A visualization of all transformations can bee seen in figure 3.



The initial configuration $\vec{X}$ of the object

The object after translation

The object after some non-uniform scaling

The object after some rotation

The object after shearing the x-coordinate with respect to the y-coordinate

Figure 3: A variety of transformations to a deformable material

In the first example, the transformation is a simple translation by some constant vector $\vec{t}$ so our deformation map $\vec{\phi}$ and deformation gradient $\mathbf{F}$ will look like

$$\begin{aligned} \vec{\phi}(\vec{\mathrm{X}}) &= \vec{\mathrm{X}} + \vec{t} \\ \mathbf{F} &= \mathbf{I}. \end{aligned} \tag{2.2}$$

Note that since we translate by some constant vector $\vec{t}$ the deformation gradient equals the identity matrix what corresponds to no deformation forces as we will see later. The situation is different if we scale the object. In our example, we squish together the material with respect to the y-coordinate and stretched it with respect to the x-coordinate. In this case $\vec{\phi}$ and $\mathbf{F}$ can be written down as

$$\begin{aligned} \vec{\phi}(\vec{\mathrm{X}}) &= \begin{pmatrix} 2 \cdot \mathrm{X}_x \\ 0.5 \cdot \mathrm{X}_y \end{pmatrix} \\ \mathbf{F} &= \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} \end{aligned} \tag{2.3}$$

so now the deformation gradient does not equal the identity matrix which is intuitively correct since scaling represents a true deformation to our object. An interesting example is rotation: When rotating our object by some angle α, this corresponds to a rigid body transformation which is not a deformation of the material. However, when writing down the deformation map and gradient of a rotation

$$\begin{aligned}\vec{\phi}(\vec{\mathrm{X}}) &= \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\mathrm{X}_x\\ \mathrm{X}_y\end{pmatrix}\\ \mathbf{F} &= \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\end{aligned} \tag{2.4}$$

we can see that even though rotations are rigid body transformations, the deformation gradient $\mathbf{F}$ is not equal the identity matrix in general. We have to keep this in mind and take care of this later on. As a last example we apply some shearing to the material, in our case we sheared the objects x-coordinate with respect to the y-coordinate. This is a true deformation of the object which is also captured in the deformation gradient:

$$\begin{aligned}\vec{\phi}(\vec{\mathrm{X}}) &= \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathrm{X}_x\\ \mathrm{X}_y\end{pmatrix}\\ \mathbf{F} &= \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.\end{aligned} \tag{2.5}$$

2.2 Strain

So far we have considered the deformation map $\vec{\phi}(\vec{\mathrm{X}})$ and its gradient $\mathbf{F}$. We can now use the deformation gradient to compute *strain*. Strain is some dimensionless description of the deformation of the material with the interesting property that it should exclude rigid body transformations. There exist many strain definitions, here we will focus on two of them: The *Green strain tensor* $\mathbf{E}$ is computed with

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^\top\mathbf{F} - \mathbf{I}\right). \tag{2.6}$$

One can see that by using the Green strain tensor, if $\mathbf{F}$ is a rotation matrix, then $\mathbf{F}^\top\mathbf{F} - \mathbf{I}$ becomes zero since $\mathbf{F}^\top = \mathbf{F}^{-1}$ for rotation matrices, so the Green strain tensor becomes zero for rotations. The same holds true for translations by a constant vector since, as we have seen in equation 2.2, the deformation gradient $\mathbf{F}$ of a translation is the identity matrix. Another interesting strain tensor is the *infinitesimal strain tensor* $\boldsymbol{\epsilon}$ which is defined as

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{F}^\top + \mathbf{F}\right) - \mathbf{I}. \tag{2.7}$$

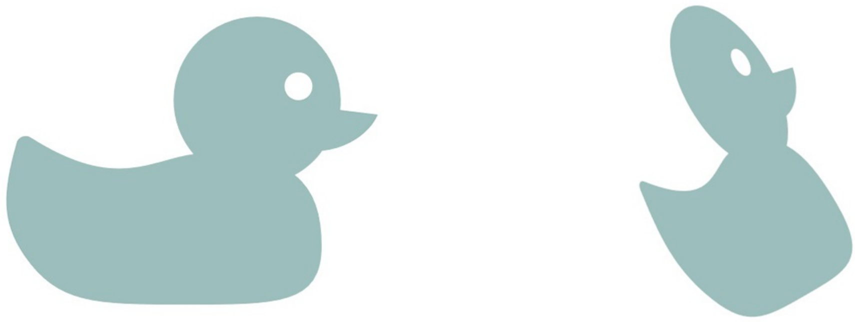
It approximates the Green strain tensor for small deformations [SB12] and since it is linearly dependent on $\mathbf{F}$ this strain tensor can be used to build a linear system for an implicit time integration scheme as done in [PGBT18]. Because rotations are reflected in the deformation gradient $\mathbf{F}$, one has to extract the non-rotation deformation out of $\mathbf{F}$ before applying the infinitesimal strain tensor.

2.3 Stress

Stress $\boldsymbol{\sigma}$ is a physical quantity that describes the internal forces in a material that all points in this material exert on each other. The relation between strain and stress, so the relation between the deformation of the object and the forces that act in the material, is given by a so called *constitutive equation*. Constitutive equations are material individual and describe their behavior caused by deformations. One example for such an constitutive equation is *Hooke's law for isotropic materials*. It relates strain and stress for linearly elastic isotropic materials by

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{(1+v)(1-2v)}\begin{pmatrix}1-v & v & v & 0 & 0 & 0\\ v & 1-v & v & 0 & 0 & 0\\ v & v & 1-v & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2}\end{pmatrix}\begin{pmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ \epsilon_{yz}\\ \epsilon_{xz}\\ \epsilon_{xy}\end{pmatrix}, \tag{2.8}$$

whereby E denotes the *Young modulus* and v Poisson's ratio. They are material parameters that can be looked up in tables. As an example, stainless steel has a Young modulus equal to $1.8 \cdot 10^{11}$Pa and a Poisson's ratio of 0.3. Contrary to that, rubber has a much smaller Young modulus of only $1.0 \cdot 10^{7}$Pa and a Poisson's ratio of 0.4999.



The initial configuration $\vec{X}$ of the duck

The current configuration $\vec{x}$ of the duck after deforming it

Figure 4: Deformation of a rubber duck

2.4 Cauchy Momentum Equation

To compute the final accelerations in the material depending on stress, we make use of the *Cauchy momentum equation* shown in equation 2.9 whereby ρ denotes the density. It relates the acceleration of a point in the material to the divergence of the stress tensor at this point and also considers external accelerations that are typically gravity or friction.

$$\vec{a} = \frac{1}{\rho}\vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{a}_{\text{external}} \tag{2.9}$$

2.5 Example

In order to improve the insight into the acceleration computation with continuum mechanics we go through an example. As we can see in figure 4, the duck is first squished together with respect to the x-axis by a factor of 2 and then rotated counterclockwise by 30°. With those information, the deformation map is given by

$$\begin{aligned}\vec{\phi}(\vec{X}) &= \begin{pmatrix}\cos 30° & -\sin 30° \\ \sin 30° & \cos 30°\end{pmatrix}\begin{pmatrix}0.5 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix}X_x \\ X_y\end{pmatrix} \\ &= \begin{pmatrix}0.433 & -0.500 \\ 0.250 & 0.866\end{pmatrix}\begin{pmatrix}X_x \\ X_y\end{pmatrix}\end{aligned} \tag{2.10}$$

and the corresponding deformation gradient then is

$$\mathbf{F} = \begin{pmatrix} 0.433 & -0.500 \\ 0.250 & 0.866 \end{pmatrix}. \tag{2.11}$$

With the deformation gradient we compute strain using the Green strain tensor:

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}\left(\mathbf{F}^\top \mathbf{F} - \mathbf{I}\right) \\ &= \begin{pmatrix} -0.375 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2.12}$$

Since the duck consists out of rubber we use Hooke's law to compute stress from strain with a Young modulus $E = 1.0 \cdot 10^7$Pa and Poisson's ratio of $v = 0.4999$

$$\begin{aligned} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} &= \frac{E}{(1+v)(1-2v)} \begin{pmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{pmatrix} \begin{pmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{xy} \end{pmatrix} \\ &= \begin{pmatrix} -6.251 \cdot 10^9 \\ -6.249 \cdot 10^9 \\ 0 \end{pmatrix} \mathrm{Pa}, \end{aligned} \tag{2.13}$$

which results in the stress tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} -6.251 \cdot 10^9 & 0 \\ 0 & -6.249 \cdot 10^9 \end{pmatrix} \mathrm{Pa}. \tag{2.14}$$

We can now use the Cauchy momentum equation to compute the final accelerations in the material. Since the accelerations are given by

$$\vec{\mathrm{a}} = \frac{1}{\rho} \vec{\nabla} \cdot \boldsymbol{\sigma} + \vec{\mathrm{a}}_{\mathrm{external}} \tag{2.15}$$

and the stress tensor is constant because it doesn't depend on the position in the material, the divergence of the stress tensor will be zero at all points in the material. Thus points in the material won't be accelerated due to deformation forces. The situation differs on the surface of the duck since at those positions there are traction forces which push the surface outwards, caused by the non-zero internal stress and zero stress on the outside.

3 Elastic Materials

We have seen concepts of continuum mechanics which we now want to apply to a concrete simulation method for elastic materials. Up until now, we considered the deformation map $\vec{\phi}$, deformation gradient $\mathbf{F}$, strain $\mathbf{E}$ and stress $\boldsymbol{\sigma}$ as continuous functions in the material space. However, computers typically can't deal with continuous representations of those fields, so we have to think of a way to discretize the material. One of the most popular discretization methods for simulating elastic materials are finite elements approaches of which we will use the method of subdividing the whole volume of the object with tetrahedrons. This way, we obtain a representation of each volume element by four vertices and the four surfaces they span.

Again, the goal is to compute accelerations at all vertices of the tetrahedrons using continuum mechanics that can be used for simulating elastic materials. In this case, we only know the initial and current vertex positions, again denoted with $\vec{\mathrm{X}}$ and $\vec{\mathrm{x}}$, and have to compute the currently unknown deformation map. The situation is illustrated in figure 5. In order to be able to compute the deformation map, we transpose the initial and the current configuration of the tetrahedron such that the vertex-position $\vec{\mathrm{X}}_0$ and $\vec{\mathrm{x}}_0$ lie on the origin of the coordinate system. Then we can compute the deformation map of the tetrahedron with

$$\vec{\phi}(\vec{\mathrm{X}}) = [\vec{\mathrm{x}}_1, \vec{\mathrm{x}}_2, \vec{\mathrm{x}}_3] \left[\vec{\mathrm{X}}_1, \vec{\mathrm{X}}_2, \vec{\mathrm{X}}_3\right]^{-1} \vec{\mathrm{X}} \tag{3.1}$$

[Tes20]. From this deformation map we can then extract the deformation gradient $\mathbf{F}$ as usual:

$$\mathbf{F} = [\vec{\mathrm{x}}_1, \vec{\mathrm{x}}_2, \vec{\mathrm{x}}_3] \left[\vec{\mathrm{X}}_1, \vec{\mathrm{X}}_2, \vec{\mathrm{X}}_3\right]^{-1} \tag{3.2}$$

This allows us to compute strain using the Green strain tensor $\mathbf{E}$ and then stress $\boldsymbol{\sigma}$ using Hooke's law. Using the stress tensor we are able to compute forces at all surface elements of the tetrahedrons with

$$\vec{f}_{ijk} = A \cdot \boldsymbol{\sigma} \cdot \vec{\mathrm{n}} \tag{3.3}$$

whereby A is the area and $\vec{\mathrm{n}}$ the normal vector of the surface element. $\boldsymbol{\sigma} \cdot \vec{\mathrm{n}}$ corresponds to the *traction* on the surface. As a last step, forces at the surface elements are equally distributed over adjacent vertices since our goal is it to compute forces at vertices. An example where we go through the simulation procedure one more time is presented in the next section.

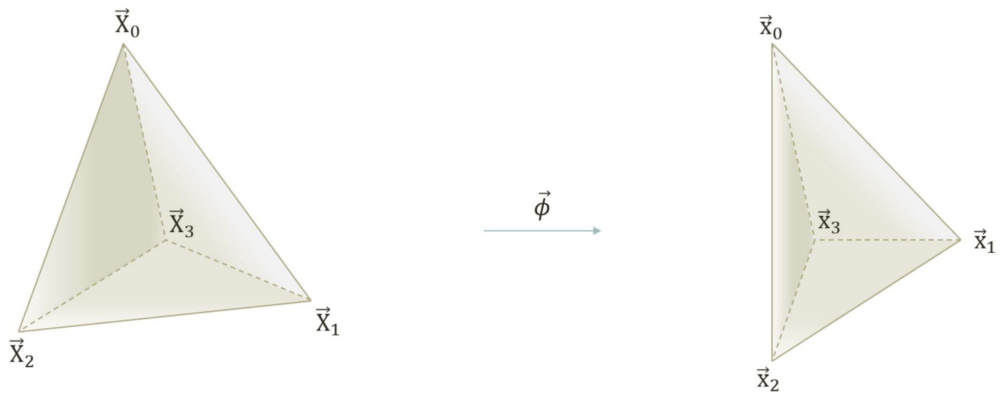


Figure 5: Deformation of a tetrahedron by a currently unknown deformation map $\vec{\phi}$

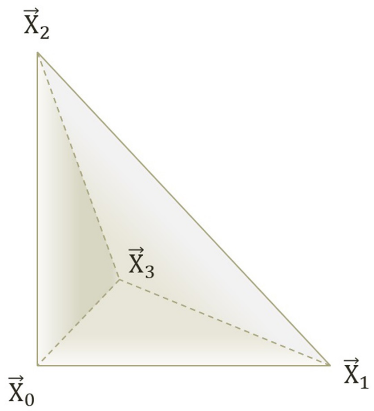
3.1 Example

In this example we will compute the elastic forces at all vertices of a tetrahedron caused by some deformation of the tetrahedron. We assume the material of the tetrahedron is rubber. An overview of the configurations of the tetrahedron and the resulting forces that will be calculated is given in figure 6. In a simulation we have only given the initial and the current vertex positions, which in our example are:

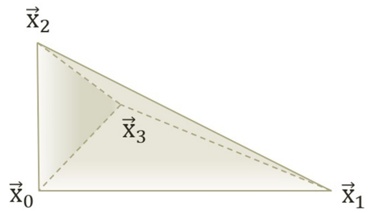
$$\begin{aligned} \vec{X}_0 &= \begin{pmatrix}1\\1\\0\end{pmatrix} & \vec{X}_1 &= \begin{pmatrix}3\\1\\0\end{pmatrix} & \vec{X}_2 &= \begin{pmatrix}1\\3\\0\end{pmatrix} & \vec{X}_3 &= \begin{pmatrix}1\\1\\1\end{pmatrix} \\ \vec{x}_0 &= \begin{pmatrix}1\\1\\0\end{pmatrix} & \vec{x}_1 &= \begin{pmatrix}3\\1\\0\end{pmatrix} & \vec{x}_2 &= \begin{pmatrix}1\\2\\0\end{pmatrix} & \vec{x}_3 &= \begin{pmatrix}1\\1\\1\end{pmatrix} \end{aligned} \tag{3.4}$$

Note that the only vertex coordinate that is changed by the deformation map is the y-coordinate of $\vec{X}_2$. We start the force calculation by computing the deformation map. To do this, we translate both configurations in a way such that $\vec{X}_0$ and $\vec{x}_0$ lie on the origin of the coordinate system. This results in the coordinates

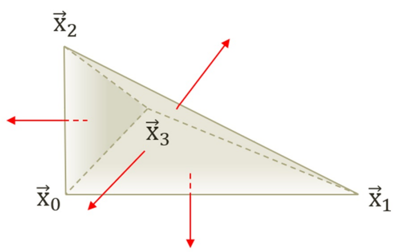
$$\begin{aligned} \vec{X}_0 &= \begin{pmatrix}0\\0\\0\end{pmatrix} & \vec{X}_1 &= \begin{pmatrix}2\\0\\0\end{pmatrix} & \vec{X}_2 &= \begin{pmatrix}0\\2\\0\end{pmatrix} & \vec{X}_3 &= \begin{pmatrix}0\\0\\1\end{pmatrix} \\ \vec{x}_0 &= \begin{pmatrix}0\\0\\0\end{pmatrix} & \vec{x}_1 &= \begin{pmatrix}2\\0\\0\end{pmatrix} & \vec{x}_2 &= \begin{pmatrix}0\\1\\0\end{pmatrix} & \vec{x}_3 &= \begin{pmatrix}0\\0\\1\end{pmatrix}. \end{aligned} \tag{3.5}$$



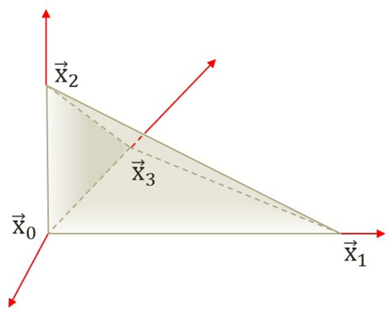
The initial configuration $\vec{X}$ of the tetrahedron



The current configuration $\vec{x}$ of the tetrahedron after some deformation



The elastic forces that act on the surfaces of the tetrahedron due to the deformation



The final forces acting at the vertices

Figure 6: Deformation and corresponding forces at the exemplary tetrahedron

Now we can use equation 3.1 to compute the deformation map $\vec{\phi}$ and deformation gradient $\mathbf{F}$:

$$\begin{aligned}
\vec{\phi}(\vec{\mathrm{X}}) &= [\vec{\mathrm{x}}_1, \vec{\mathrm{x}}_2, \vec{\mathrm{x}}_3] \left[\vec{\mathrm{X}}_1, \vec{\mathrm{X}}_2, \vec{\mathrm{X}}_3\right]^{-1} \vec{\mathrm{X}} \\
&= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \vec{\mathrm{X}} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\mathrm{X}}
\end{aligned} \tag{3.6}$$

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As already done earlier, we use the deformation gradient to compute the Green strain tensor $\mathbf{E}$ whose definition was presented in equation 2.6

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^\top \mathbf{F} - \mathbf{I}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -0.375 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

and plug the entries of the strain tensor in Hooke's law shown in equation 2.8. Together with the Young modulus and Poisson's ratio of rubber ($E = 10^7 \mathrm{Pa}, v = 0.4999$) this gives us the stress tensor

$$\boldsymbol{\sigma} = \begin{pmatrix} -6.249 \cdot 10^9 & 0 & 0 \\ 0 & -6.251 \cdot 10^9 & 0 \\ 0 & 0 & -6.249 \cdot 10^9 \end{pmatrix} \mathrm{Pa}. \tag{3.8}$$

The stress tensor is used to compute the forces that act on each surface element using equation 3.3, which results in the following four surface forces, assuming that the vertex positions are given in meters:

$$
\begin{aligned}
\vec{f}_{012} &= 1\text{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6.249 \cdot 10^9 \end{pmatrix} \text{N} \\
\vec{f}_{013} &= 1\text{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -6.251 \cdot 10^9 \\ 0 \end{pmatrix} \text{N} \\
\vec{f}_{023} &= 0.5\text{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3.125 \cdot 10^9 \\ 0 \\ 0 \end{pmatrix} \text{N} \\
\vec{f}_{123} &= \frac{3}{2}\text{m}^2 \cdot \boldsymbol{\sigma} \cdot \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -3.125 \cdot 10^9 \\ -6.251 \cdot 10^9 \\ -6.249 \cdot 10^9 \end{pmatrix} \text{N}
\end{aligned}
\tag{3.9}
$$

An illustration of the surface forces is presented in figure 6, note how the surface elements are pushed outwards as an effect of the compression of the tetrahedron. As a last step, we equally distribute the surface forces on adjacent vertices to obtain the final forces at the vertices:

$$
\begin{aligned}
\vec{f}(\vec{\mathrm{x}}_0) &= \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{013} + \vec{f}_{023}\right) = \begin{pmatrix} -1.04 \\ -2.08 \\ -2.08 \end{pmatrix} \cdot 10^9\text{N} \\
\vec{f}(\vec{\mathrm{x}}_1) &= \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{013} + \vec{f}_{123}\right) = \begin{pmatrix} 1.04 \\ 0 \\ 0 \end{pmatrix} \cdot 10^9\text{N} \\
\vec{f}(\vec{\mathrm{x}}_2) &= \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 2.08 \\ 0 \end{pmatrix} \cdot 10^9\text{N} \\
\vec{f}(\vec{\mathrm{x}}_2) &= \frac{1}{3}\left(\vec{f}_{012} + \vec{f}_{023} + \vec{f}_{123}\right) = \begin{pmatrix} 0 \\ 0 \\ 2.08 \end{pmatrix} \cdot 10^9\text{N}
\end{aligned}
\tag{3.10}
$$

One interesting property which can be seen here is since all forces of the vertices add up to zero, they conserve the momentum of the tetrahedron. Those final forces at the vertices can now be used in a simulation to perform a time integration step and thus simulate the behavior of the elastic material sampled by the tetrahedrons.

4 Summary

We have seen that continuum mechanics have many real-life applications. Among other reasons, this is due to the fact that, while we have a similar procedure for all materials, the flexibility of constitutive equations allows the representation of a wide range of individual material behaviors. If we want to simulate materials using continuum mechanics we have to think of a discretization method since computers can't deal with continuous functions. In our example we have seen a finite elements approach but there exist a large variety of possible disretization schemes that one could use, such as the Material point method or Smoothed Particle Hydrodynamics.

References

[GAR18] Daniel George, Rachele Allena, and Yves Remond. A multiphysics stimulus for continuum mechanics bone remodeling. *Mathematics and Mechanics of Complex Systems*, 6(4):307–319, 2018.

[PGBT18] Andreas Peer, Christoph Gissler, Stefan Band, and Matthias Teschner. An implicit SPH formulation for incompressible linearly elastic solids. In *Computer Graphics Forum*, volume 37, pages 135–148. Wiley Online Library, 2018.

[RPC+19] Monika Ratajczak, Mariusz Ptak, Leszek Chybowski, Katarzyna Gawdzińska, and Romuald Będziński. Material and structural modeling aspects of brain tissue deformation under dynamic loads. *Materials*, 12(2):271, 2019.

[SB12] Eftychios Sifakis and Jernej Barbic. FEM simulation of 3d deformable solids: a practitioner's guide to theory, discretization and model reduction. In *Acm siggraph 2012 courses*, pages 1–50. 2012.

[Tes20] Matthias Teschner. Simulation in computer graphics exercises - notes. 2020.