Simulation in Computer Graphics
Elastic Solids 1

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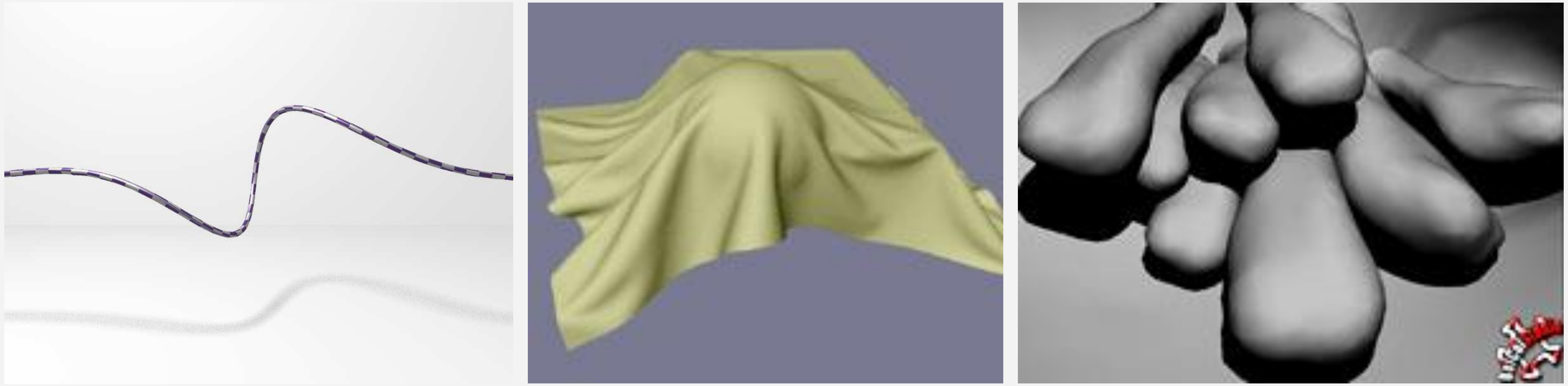
Outline

- Introduction
- Elastic forces
- Miscellaneous
- Collision handling
- Visualization

Motivation

- Elastic solids are modeled for particle sets, i.e. elements
- Forces at particles account for resistance to deformation, e.g., stretch, shear, bend, volume change



Elastic solids in 1D, 2D, and 3D

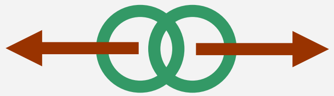
Motivation

- Element in rest state / undeformed state
  ⇨ No elastic forces
- Element in deformed state
  ⇨ Elastic forces that accelerate particles towards the rest state of an element

Motivation

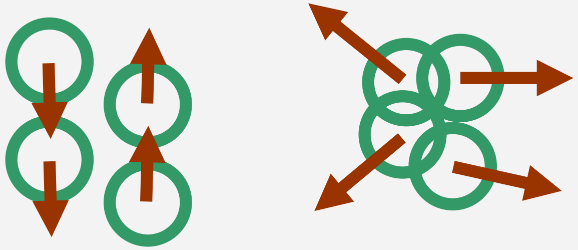
- Different types of deformation (degrees of freedom) can be derived from the incorporated particles
  - Two particles: stretch, compression
  - Three particles: area, shear
  - More particles: volume, shear

Rest state



Deformed state with elastic forces

Rest state



Deformed states with elastic forces

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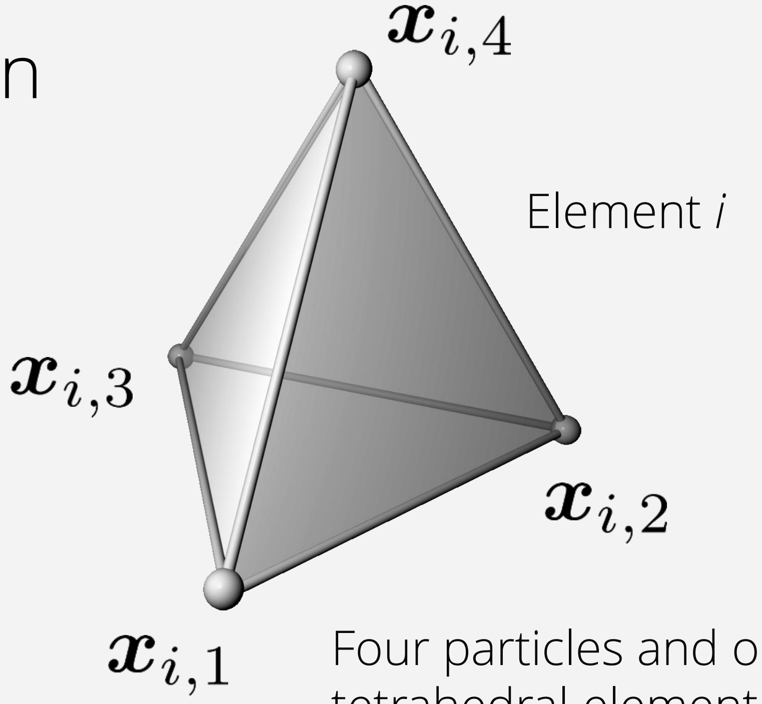
Overview

- Define elements
- Compute deformation / strain
- Compute stress
- Compute elastic energy
- Compute elastic forces

Elements

- Particles form elements, e.g.
  - Two particles form a line segment
  - Four particles build a tetrahedron
  - …



Four particles and one tetrahedral element

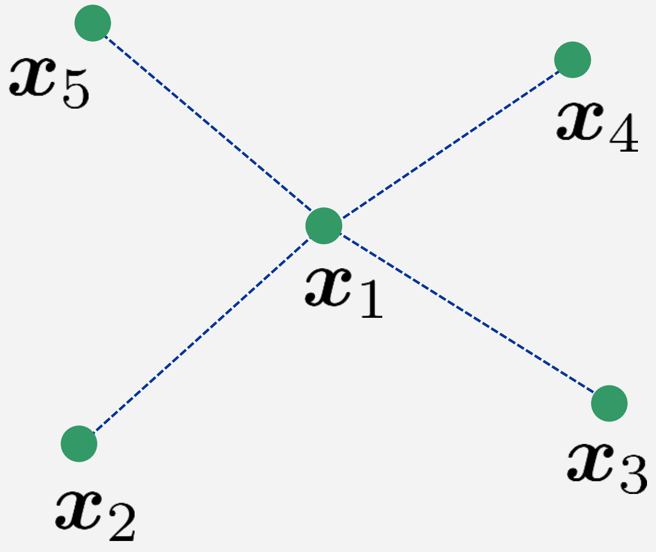
Deformation / Strain

- Define a function that describes a deformation of element i based on particle positions: $C_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n})$
- Undeformed state: $C_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) = 0$
- Deformed state: $C_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) \neq 0$
- Example: relative stretching / compressing of two points $\boldsymbol{x}_1, \boldsymbol{x}_2$ with rest distance L_i:

$$C_i(\boldsymbol{x}_{i,1}, \boldsymbol{x}_{i,2}) = \frac{1}{L_i}(|\boldsymbol{x}_{i,1} - \boldsymbol{x}_{i,2}| - L_i)$$

Strain is dimensionless.

Deformation of Line Segments



Five particles and
four elements

$$C_1(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{L_1}(|\boldsymbol{x}_1 - \boldsymbol{x}_2| - L_1)$$

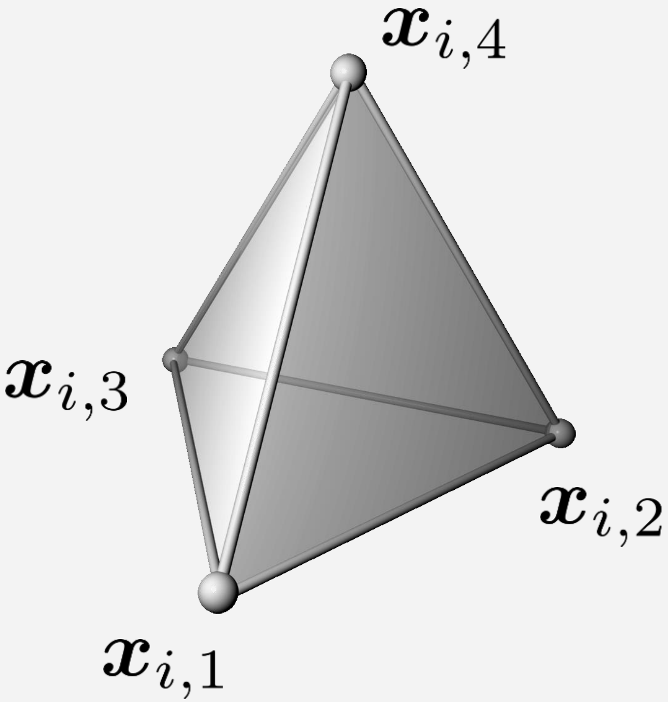
$$C_2(\boldsymbol{x}_1, \boldsymbol{x}_3) = \frac{1}{L_2}(|\boldsymbol{x}_1 - \boldsymbol{x}_3| - L_2)$$

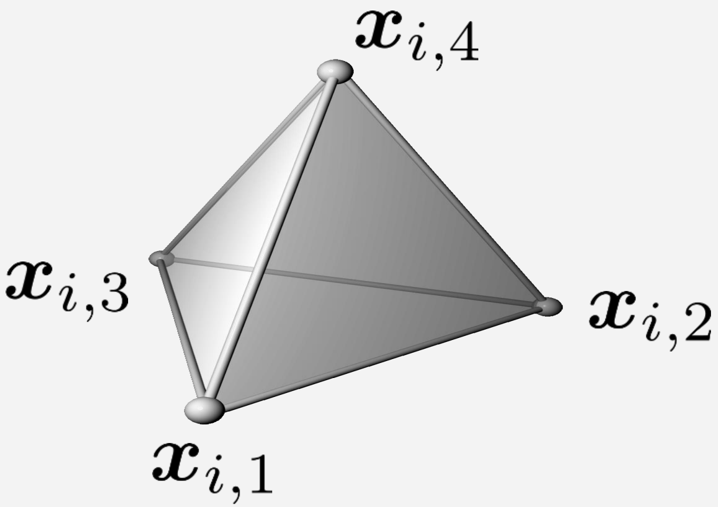
$$C_3(\boldsymbol{x}_1, \boldsymbol{x}_4) = \frac{1}{L_3}(|\boldsymbol{x}_1 - \boldsymbol{x}_4| - L_3)$$

$$C_4(\boldsymbol{x}_1, \boldsymbol{x}_5) = \frac{1}{L_4}(|\boldsymbol{x}_1 - \boldsymbol{x}_5| - L_4)$$

Exemplary deformation
computations for the elements

Deformation of a Tetrahedron



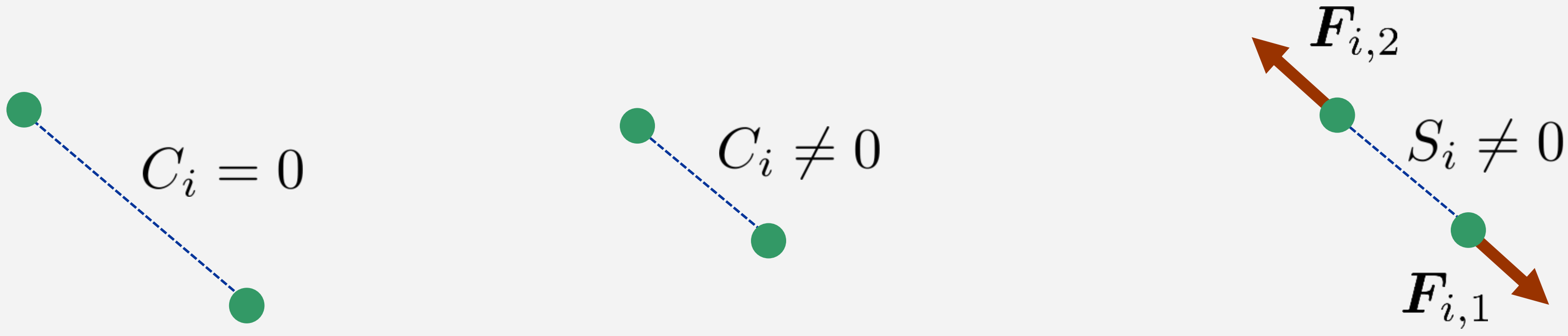


$$C_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,4}) = 0$$

$$C_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,4}) \neq 0$$

Stress

- Deformation of an element causes element stress
- Internal pressure (force per area)
- $S_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) = k_i \; C_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n})$
  - Material stiffness k_i



Rest state

Deformed state

Stress due to deformation.
Forces due to stress.

Elastic Energy

- Work that is performed to deform an element is stored as elastic energy:
  - $E_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) = \frac{1}{2}\ S_i\ C_i\ V_i = \frac{1}{2}\ k_i\ C_i^2\ V_i$
  - V_i is the size (volume / area / length) of an element
- Quantifies the deformation of an element
  - Undeformed state: $E_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) = 0$
  - Deformed state: $E_i(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) > 0$

Elastic Forces

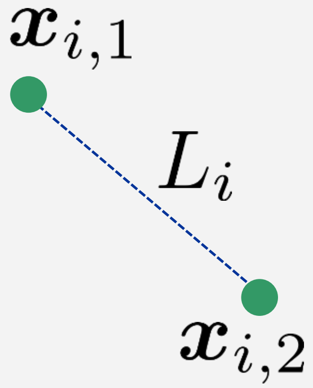
- Accelerate particles from positions with high elastic energy towards positions with low elastic energy
  - Negative spatial gradient of the elastic energy
- Goal: Minimization of elastic energy, i.e. deformation
- For all particles $1 \leq j \leq n$ of an element i:

$$\begin{aligned}\boldsymbol{F}_{i,j}(\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,n}) &= -\frac{\partial}{\partial \boldsymbol{x}_{i,j}} E_i(\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,n}) \\ &= -k_i V_i C_i(\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,n}) \frac{\partial}{\partial \boldsymbol{x}_{i,j}} C_i(\boldsymbol{x}_{i,1},\ldots,\boldsymbol{x}_{i,n})\end{aligned}$$

Distance Change

- Two particles $\boldsymbol{x}_{i,1}, \boldsymbol{x}_{i,2}$ form an element i

$$\boldsymbol{x}_{i,1} = \begin{pmatrix} x_{i,1} \\ y_{i,1} \\ z_{i,1} \end{pmatrix} \quad \boldsymbol{x}_{i,2} = \begin{pmatrix} x_{i,2} \\ y_{i,2} \\ z_{i,2} \end{pmatrix}$$



- Strain

$$C_i^{\mathrm{d}}(\boldsymbol{x}_{i,1}, \boldsymbol{x}_{i,2}) = \frac{1}{L_i}(|\boldsymbol{x}_{i,1} - \boldsymbol{x}_{i,2}| - L_i)$$

$$C_i^{\mathrm{d}}(x_{i,1}, y_{i,1}, z_{i,1}, x_{i,2}, z_{i,2}, z_{i,2})$$
$$= \frac{1}{L_i}(\sqrt{(x_{i,1} - x_{i,2})^2 + (y_{i,1} - y_{i,2})^2 + (z_{i,1} - z_{i,2})^2} - L_i)$$

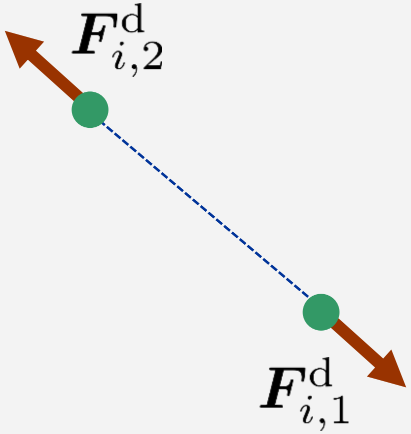
Distance Change

- Spatial derivatives of the strain

$$\frac{\partial C_i^{\mathrm{d}}}{\partial \boldsymbol{x}_{i,1}} = \begin{pmatrix} \frac{\partial C_i^{\mathrm{d}}}{\partial x_{i,1}} \\ \frac{\partial C_i^{\mathrm{d}}}{\partial y_{i,1}} \\ \frac{\partial C_i^{\mathrm{d}}}{\partial z_{i,1}} \end{pmatrix} = \frac{1}{L_i |\boldsymbol{x}_{i,1} - \boldsymbol{x}_{i,2}|} \begin{pmatrix} x_{i,1} - x_{i,2} \\ y_{i,1} - y_{i,2} \\ z_{i,1} - z_{i,2} \end{pmatrix} = \frac{\boldsymbol{x}_{i,1} - \boldsymbol{x}_{i,2}}{L_i |\boldsymbol{x}_{i,1} - \boldsymbol{x}_{i,2}|}$$

$$\frac{\partial C_i^{\mathrm{d}}}{\partial \boldsymbol{x}_{i,2}} = -\frac{\partial C_i^{\mathrm{d}}}{\partial \boldsymbol{x}_{i,1}}$$

Distance Change



- Elastic force at particle $\boldsymbol{x}_{i,1}$:

$$\boldsymbol{F}_{i,1}^{\mathrm{d}}(\boldsymbol{x}_{i,1},\boldsymbol{x}_{i,2}) = -k_i^{\mathrm{d}}\ L_i\ C_i^{\mathrm{d}}\ \frac{\partial C_i^{\mathrm{d}}}{\partial \boldsymbol{x}_{i,1}}$$
$$= -k_i^{\mathrm{d}} \frac{|\boldsymbol{x}_{i,1}-\boldsymbol{x}_{i,2}|-L_i}{L_i} \frac{\boldsymbol{x}_{i,1}-\boldsymbol{x}_{i,2}}{|\boldsymbol{x}_{i,1}-\boldsymbol{x}_{i,2}|}$$

- Elastic force at particle $\boldsymbol{x}_{i,2}$:

$$\boldsymbol{F}_{i,2}^{\mathrm{d}}(\boldsymbol{x}_{i,1},\boldsymbol{x}_{i,2}) = -k_i^{\mathrm{d}}\ L_i\ C_i^{\mathrm{d}}\ \frac{\partial C_i^{\mathrm{d}}}{\partial \boldsymbol{x}_{i,2}}$$
$$= k_i^{\mathrm{d}} \frac{|\boldsymbol{x}_{i,1}-\boldsymbol{x}_{i,2}|-L_i}{L_i} \frac{\boldsymbol{x}_{i,1}-\boldsymbol{x}_{i,2}}{|\boldsymbol{x}_{i,1}-\boldsymbol{x}_{i,2}|}$$

Area Change

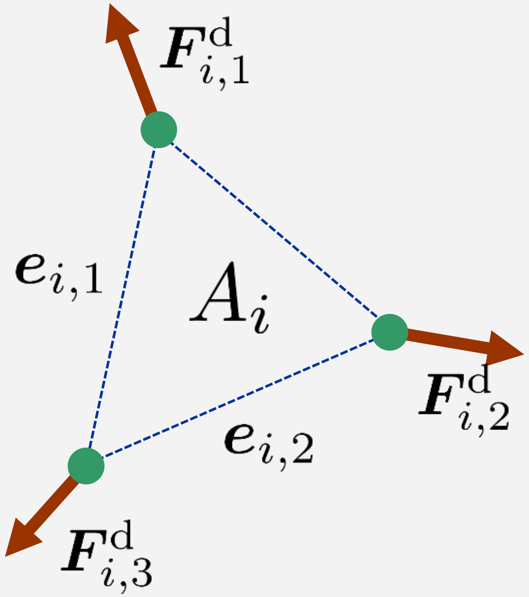
- Three particles $\boldsymbol{x}_{i,1}, \boldsymbol{x}_{i,2}, \boldsymbol{x}_{i,3}$ form a triangle i
- Edges: $\boldsymbol{e}_{i,1} = \boldsymbol{x}_{i,3} - \boldsymbol{x}_{i,1}, \boldsymbol{e}_{i,2} = \boldsymbol{x}_{i,3} - \boldsymbol{x}_{i,2}$
- Strain: $C^{\mathrm{a}} = \frac{1}{A_i}(\frac{1}{2}|\boldsymbol{e}_{i,1} \times \boldsymbol{e}_{i,2}| - A_i)$
- Forces: $\boldsymbol{F}^{\mathrm{a}}_{i,1} = s_i \; \boldsymbol{e}_{i,2} \times \boldsymbol{t}_i$

$$\boldsymbol{F}^{\mathrm{a}}_{i,2} = s_i \; \boldsymbol{t}_i \times \boldsymbol{e}_{i,1}$$

$$\boldsymbol{F}^{\mathrm{a}}_{i,3} = s_i \; \boldsymbol{t}_i \times (\boldsymbol{e}_{i,2} - \boldsymbol{e}_{i,1})$$

$$s_i = k^{\mathrm{a}}_i \frac{C^{\mathrm{a}}_i}{2|\boldsymbol{e}_{i,1} \times \boldsymbol{e}_{i,2}|}$$

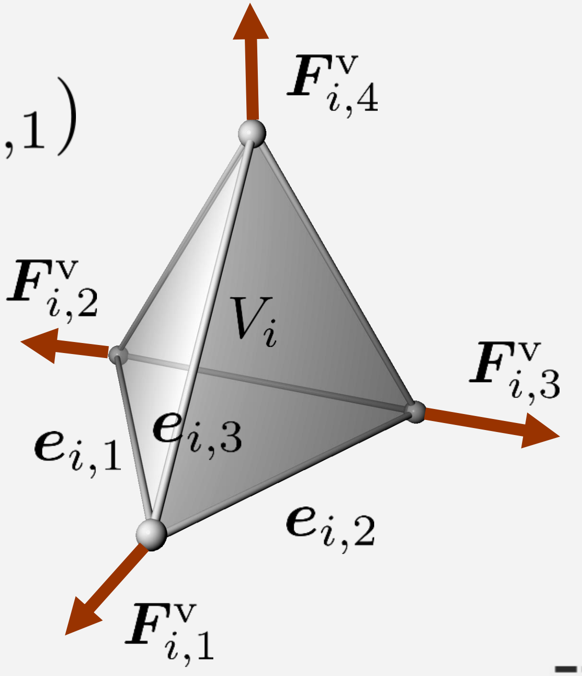
$$\boldsymbol{t}_i = \boldsymbol{e}_{i,1} \times \boldsymbol{e}_{i,2}$$



3D Volume Change

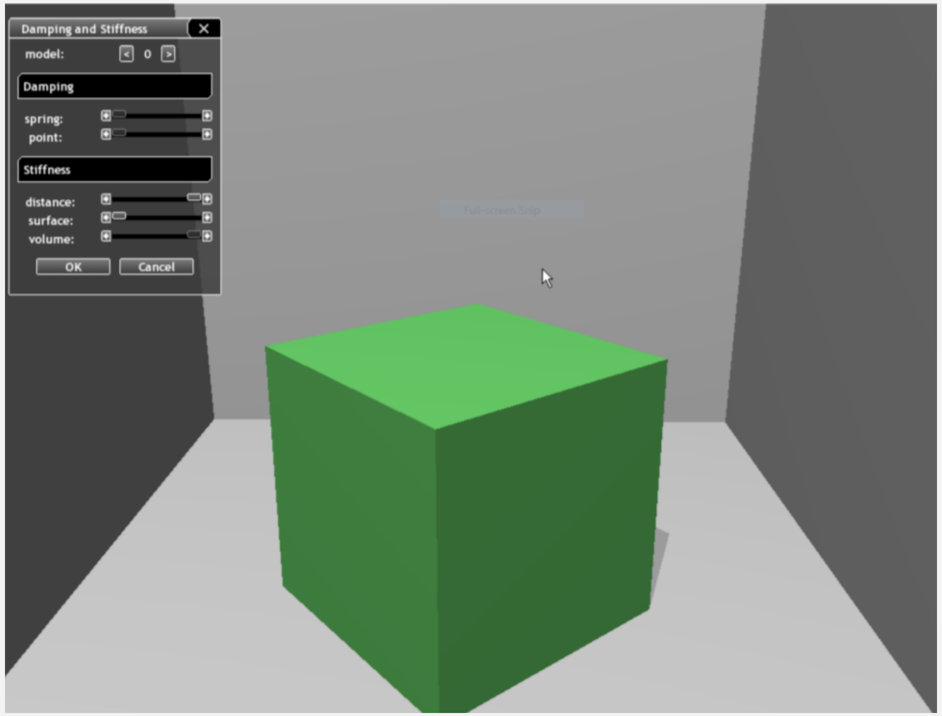
- Four particles $\boldsymbol{x}_{i,1}, \boldsymbol{x}_{i,2}, \boldsymbol{x}_{i,3}, \boldsymbol{x}_{i,4}$ form a tetrahedron i
- Edges: $\boldsymbol{e}_{i,1} = \boldsymbol{x}_{i,2} - \boldsymbol{x}_{i,1}, \boldsymbol{e}_{i,2} = \boldsymbol{x}_{i,3} - \boldsymbol{x}_{i,1}, \boldsymbol{e}_{i,3} = \boldsymbol{x}_{i,4} - \boldsymbol{x}_{i,1}$
- Strain: $C^{\mathrm{v}} = \frac{1}{V_i}(\frac{1}{6}\boldsymbol{e}_{i,1}(\boldsymbol{e}_{i,2} \times \boldsymbol{e}_{i,3}) - V_i)$
- Forces

$$\begin{aligned}
\boldsymbol{F}^{\mathrm{v}}_{i,1} &= \tfrac{1}{6}k^{\mathrm{v}}_i C^{\mathrm{v}}_i (\boldsymbol{e}_{i,2} - \boldsymbol{e}_{i,1}) \times (\boldsymbol{e}_{i,3} - \boldsymbol{e}_{i,1}) \\
\boldsymbol{F}^{\mathrm{v}}_{i,2} &= \tfrac{1}{6}k^{\mathrm{v}}_i C^{\mathrm{v}}_i \ \boldsymbol{e}_{i,3} \times \boldsymbol{e}_{i,2} \\
\boldsymbol{F}^{\mathrm{v}}_{i,3} &= \tfrac{1}{6}k^{\mathrm{v}}_i C^{\mathrm{v}}_i \ \boldsymbol{e}_{i,1} \times \boldsymbol{e}_{i,3} \\
\boldsymbol{F}^{\mathrm{v}}_{i,4} &= \tfrac{1}{6}k^{\mathrm{v}}_i C^{\mathrm{v}}_i \ \boldsymbol{e}_{i,2} \times \boldsymbol{e}_{i,1}
\end{aligned}$$



Demo

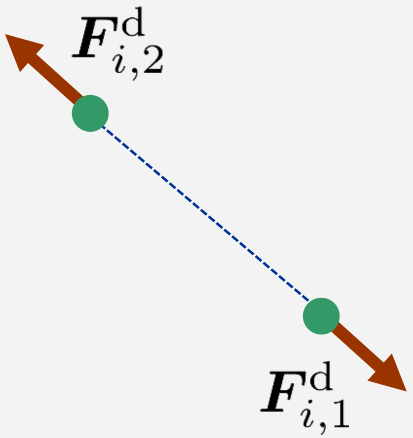
- Tetrahedral mesh with combined strain function $C^{\mathrm{d}}, C^{\mathrm{a}}, C^{\mathrm{v}}$



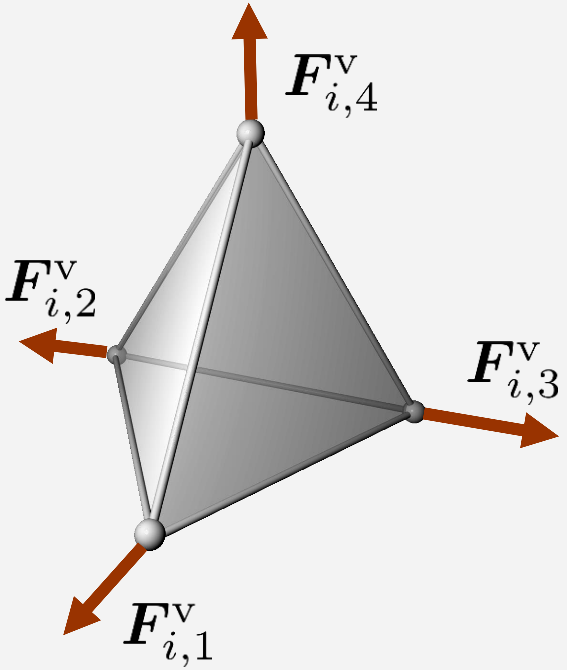
model: 0

Properties of Elastic Forces

- Preserve linear and angular momentum of an element or of a system of elements



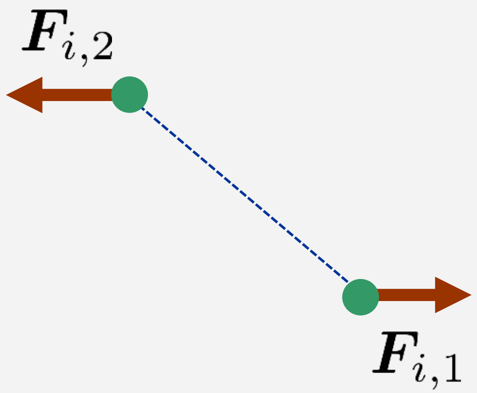
Distance forces change the distance between the two points, but not the linear and angular velocity of the spring



Volume forces change the volume of the tetrahedron, but not its velocity.

Properties of Elastic Forces

- Sum up to zero for all particles
  - $\sum_{j=1..n} \boldsymbol{F}_{i,j}(\boldsymbol{x}_{i,1}, \ldots, \boldsymbol{x}_{i,n}) = 0$
  - Do not change linear momentum
- Do not cause torque
  - Do not change angular momentum
- Also referred to as internal forces
  - External forces can change linear and angular momentum of an element, e.g. gravitational force



Forces sum up to zero, but change angular momentum of the element
⇨ no elastic force

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Damping

- Improves the stability of a particle system in explicit integration schemes
  - Typically omitted in implicit schemes with artificial damping
- Models friction or viscosity
- Force proportional to a velocity
- Directed in the opposite direction of a velocity

Damping

- Particle velocity

$$\boldsymbol{F}_i^{\text{damp}} = -\gamma \; \boldsymbol{v}_i$$

Damping
parameter

- Relative velocity

$$\boldsymbol{F}_{i,j}^{\text{damp}} = \gamma \; \left((\boldsymbol{v}_j - \boldsymbol{v}_i) \frac{\boldsymbol{x}_j - \boldsymbol{x}_i}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|} \right) \frac{\boldsymbol{x}_j - \boldsymbol{x}_i}{\|\boldsymbol{x}_j - \boldsymbol{x}_i\|}$$

Damping
parameter

Relative velocity
projected onto
direction $x_j - x_i$

Normalized
direction

Damping

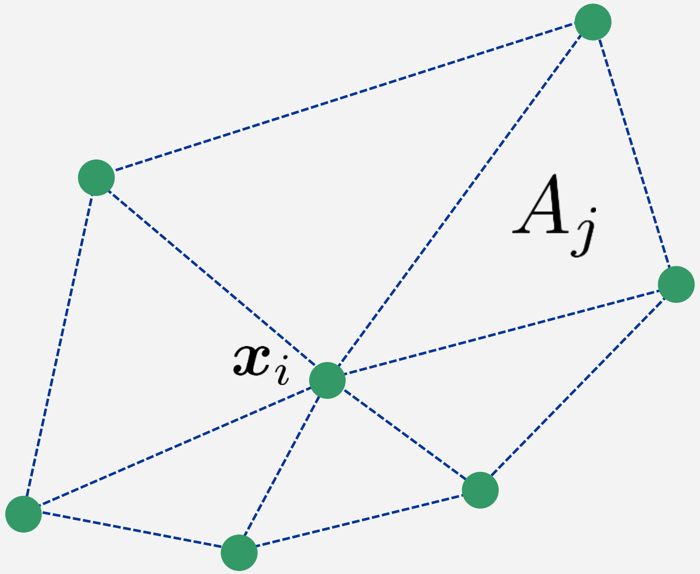
- Particle velocity
  - External force
  - Affects the global dynamics of a particle system, i.e. slows it down
- Relative velocity
  - Internal force
  - Does not affect linear and angular momentum of a particle system
  - Reduces oscillations / noise

Particle Masses

- Should be proportional to the particle size
  - Discretization should not affect the simulation
- E.g., $m_i = \underbrace{\rho_i}_{\text{Mass density of the material}} \; \frac{1}{3} \underbrace{\sum_j A_j}_{\text{Particle size}}$
- Elastic accelerations are accumulated at particles

$$\boldsymbol{a}_i = \frac{1}{\rho_i \left(\frac{1}{3}\sum_j A_j\right)} \sum_j \boldsymbol{F}_{j,i} \quad \text{or}$$

$$\boldsymbol{a}_i = \sum_j \frac{\boldsymbol{F}_{j,i}}{\rho_j \frac{1}{3} A_j}$$



Time Step

- An element should not move farther than its size in one simulation step, e.g. its diameter d: $h|\boldsymbol{v}| \leq d$
- Time step limit: $h \leq \frac{d}{|\boldsymbol{v}|}$
- $h = \lambda \frac{d}{|\boldsymbol{v}|}$ with $0 < \lambda \leq 1$
- $\lambda = \frac{h|\boldsymbol{v}|}{d}$ can be interpreted as performance measure
- Time step size is only meaningful if related to the element size

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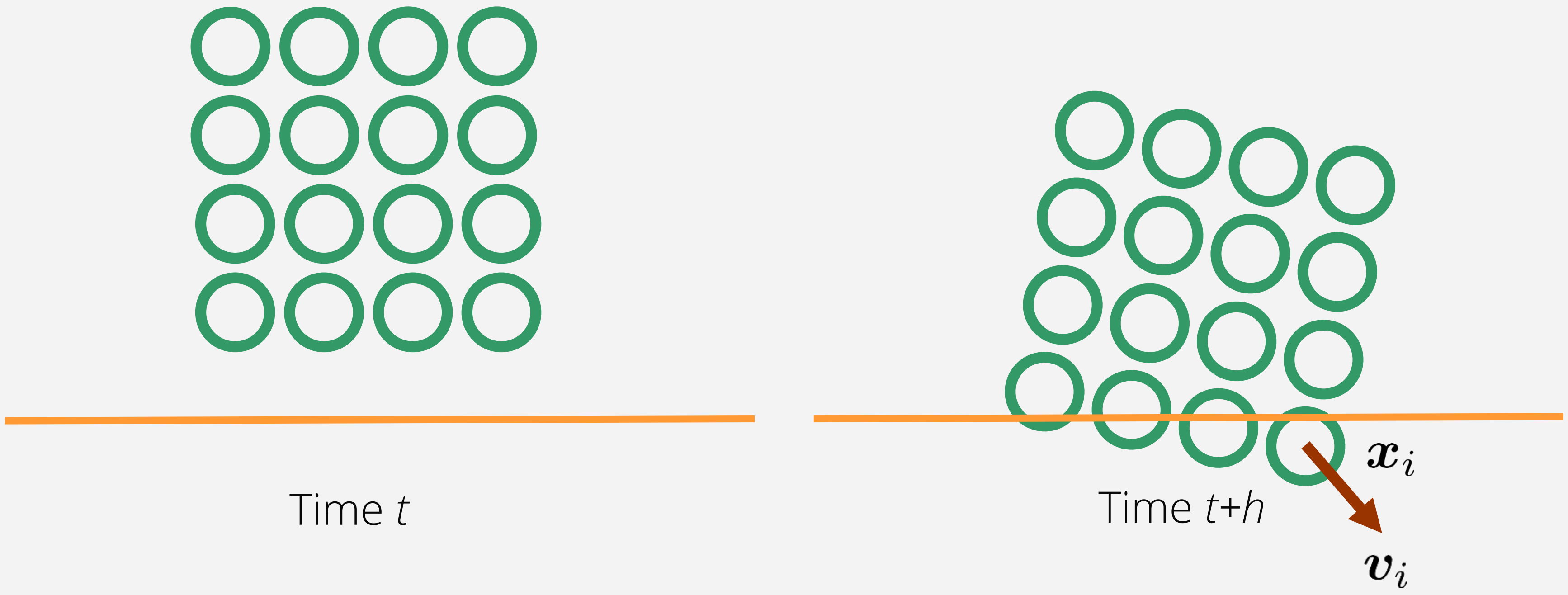
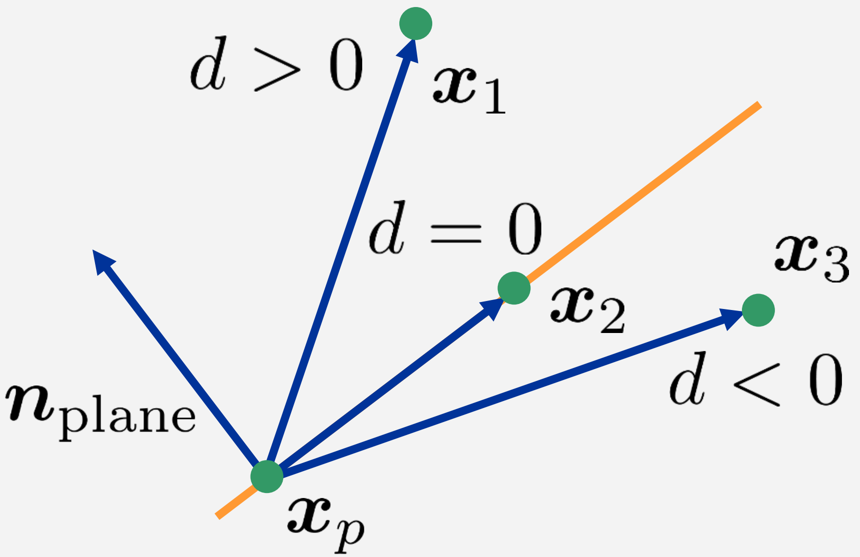
Context

- Collision of a particle of an elastic solid with a plane



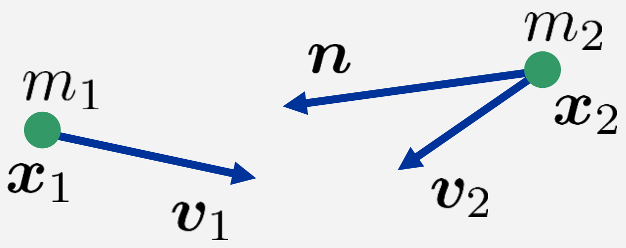
Plane Representation

- In 3D, a plane can be defined with a point $\boldsymbol{x}_p$ on the plane and a normalized plane normal $\boldsymbol{n}_{\text{plane}}$
- The plane is the set of points $\boldsymbol{x}$ with $\boldsymbol{n}_{\text{plane}} \cdot (\boldsymbol{x} - \boldsymbol{x}_p) = 0$
- For a point $\boldsymbol{x}$, the distance to the plane is $d = \boldsymbol{n}_{\text{plane}} \cdot (\boldsymbol{x} - \boldsymbol{x}_p)$



Concept

- If a collision is detected, i.e. $d < 0$, a collision impulse is computed that prevents the interpenetration of the mass point and the plane
- We first consider the case of a particle-particle collision with $\boldsymbol{n}$ being the normalized direction from $\boldsymbol{x}_2$ to $\boldsymbol{x}_1$



- The response scheme is later adapted to the particle-plane case

Coordinate Systems

- Velocities $\boldsymbol{v}$ before the collision response and $\boldsymbol{V}$ velocities after the collision response are considered in the coordinate system defined by collision normal $\boldsymbol{n}$ and two orthogonal normalized tangent axes $\boldsymbol{t}$ and $\boldsymbol{k}$
- E.g. $$\begin{pmatrix} v_{1,n} \\ v_{1,t} \\ v_{1,k} \end{pmatrix} = \begin{pmatrix} n_x & n_y & n_z \\ t_x & t_y & t_z \\ k_x & k_y & k_z \end{pmatrix} \begin{pmatrix} v_{1,x} \\ v_{1,y} \\ v_{1,z} \end{pmatrix}$$
- The velocity $\boldsymbol{V}$ after the response is transformed back $$\begin{pmatrix} V_{1,x} \\ V_{1,y} \\ V_{1,z} \end{pmatrix} = \begin{pmatrix} n_x & t_x & k_x \\ n_y & t_y & k_y \\ n_z & t_z & k_z \end{pmatrix} \begin{pmatrix} V_{1,n} \\ V_{1,t} \\ V_{1,k} \end{pmatrix}$$

Governing Equations

- Conservation of momentum

$$\begin{aligned} m_1 V_{1,n} - m_1 v_{1,n} &= P_n & m_2 V_{2,n} - m_2 v_{2,n} &= -P_n \\ m_1 V_{1,t} - m_1 v_{1,t} &= P_t & m_2 V_{2,t} - m_2 v_{2,t} &= -P_t \\ m_1 V_{1,k} - m_1 v_{1,k} &= P_k & m_2 V_{2,k} - m_2 v_{2,k} &= -P_k \end{aligned}$$

- Coefficient of restitution, $e = 1$ elastic, $e = 0$ inelastic

$$V_{1,n} - V_{2,n} = -e(v_{1,n} - v_{2,n})$$

- Friction opposes sliding motion along $\boldsymbol{t}$ and $\boldsymbol{k}$

$$P_t = \mu P_n \qquad P_k = \mu P_n$$

Linear System

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ -\mu & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mu & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & m_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & m_2 \end{pmatrix} \begin{pmatrix} P_n \\ P_k \\ P_t \\ V_{1,n} \\ V_{1,t} \\ V_{1,k} \\ V_{2,n} \\ V_{2,t} \\ V_{2,k} \end{pmatrix} = \begin{pmatrix} -e(v_{1,n} - v_{2,n}) \\ 0 \\ 0 \\ m_1 v_{1,n} \\ m_1 v_{1,t} \\ m_1 v_{1,k} \\ m_2 v_{2,n} \\ m_2 v_{2,t} \\ m_2 v_{2,k} \end{pmatrix}$$

Solution

$$\begin{pmatrix}
\frac{m_1 m_2}{m_1+m_2} & 0 & 0 & -\frac{m_2}{m_1+m_2} & 0 & 0 & \frac{m_1}{m_1+m_2} & 0 & 0 \\
\frac{m_1 m_2 \mu}{m_1+m_2} & 1 & 0 & -\frac{m_2 \mu}{m_1+m_2} & 0 & 0 & \frac{m_1 \mu}{m_1+m_2} & 0 & 0 \\
\frac{m_1 m_2 \mu}{m_1+m_2} & 0 & 1 & -\frac{m_2 \mu}{m_1+m_2} & 0 & 0 & \frac{m_1 \mu}{m_1+m_2} & 0 & 0 \\
\frac{m_2}{m_1+m_2} & 0 & 0 & \frac{1}{m_1+m_2} & 0 & 0 & \frac{1}{m_1+m_2} & 0 & 0 \\
\frac{m_2 \mu}{m_1+m_2} & \frac{1}{m_1} & 0 & -\frac{m_2 \mu}{m_1(m_1+m_2)} & \frac{1}{m_1} & 0 & \frac{\mu}{m_1+m_2} & 0 & 0 \\
\frac{m_2 \mu}{m_1+m_2} & 0 & \frac{1}{m_1} & -\frac{m_2 \mu}{m_1(m_1+m_2)} & 0 & \frac{1}{m_1} & \frac{\mu}{m_1+m_2} & 0 & 0 \\
-\frac{m_1}{m_1+m_2} & 0 & 0 & \frac{1}{m_1+m_2} & 0 & 0 & \frac{1}{m_1+m_2} & 0 & 0 \\
-\frac{m_1 \mu}{m_1+m_2} & -\frac{1}{m_2} & 0 & \frac{\mu}{m_1+m_2} & 0 & 0 & -\frac{m_1 \mu}{m_2(m_1+m_2)} & \frac{1}{m_2} & 0 \\
-\frac{m_1 \mu}{m_1+m_2} & 0 & -\frac{1}{m_2} & \frac{\mu}{m_1+m_2} & 0 & 0 & -\frac{m_1 \mu}{m_2(m_1+m_2)} & 0 & \frac{1}{m_2}
\end{pmatrix}
\begin{pmatrix}
-e(v_{1,n} - v_{2,n}) \\ 0 \\ 0 \\ m_1 v_{1,n} \\ m_1 v_{1,t} \\ m_1 v_{1,k} \\ m_2 v_{2,n} \\ m_2 v_{2,t} \\ m_2 v_{2,k}
\end{pmatrix}
=
\begin{pmatrix}
P_n \\ P_k \\ P_t \\ V_{1,n} \\ V_{1,t} \\ V_{1,k} \\ V_{2,n} \\ V_{2,t} \\ V_{2,k}
\end{pmatrix}$$

Particle ⇨ Plane

- Plane has infinite mass and does not move: $\boldsymbol{v}_2 = \boldsymbol{V}_2 = 0$
- Columns 2, 3, 7, 8, 9 do not contribute to the solution
- To solve for the particle velocity $\boldsymbol{V}_1$ after collision response, rows 4, 5, 6 have to be considered

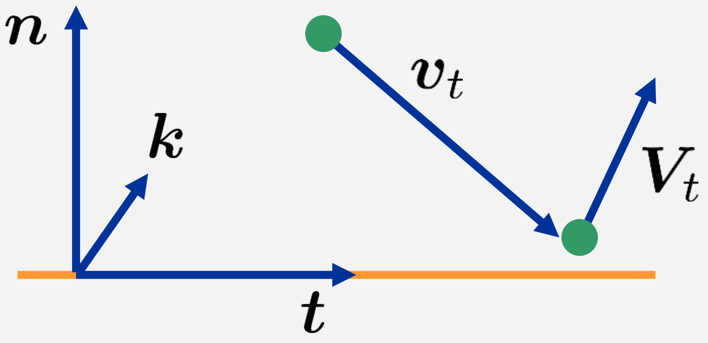
$$\begin{pmatrix} \frac{m_2}{m_1+m_2} & \frac{1}{m_1+m_2} & 0 & 0 \\ \frac{m_2\mu}{m_1+m_2} & -\frac{m_2\mu}{m_1(m_1+m_2)} & \frac{1}{m_1} & 0 \\ \frac{m_2\mu}{m_1+m_2} & -\frac{m_2\mu}{m_1(m_1+m_2)} & 0 & \frac{1}{m_1} \end{pmatrix} \begin{pmatrix} -ev_{1,n} \\ m_1 v_{1,n} \\ m_1 v_{1,t} \\ m_1 v_{1,k} \end{pmatrix} = \begin{pmatrix} V_{1,n} \\ V_{1,t} \\ V_{1,k} \end{pmatrix}$$

- Plane has infinite mass

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mu & -\frac{\mu}{m_1} & \frac{1}{m_1} & 0 \\ \mu & -\frac{\mu}{m_1} & 0 & \frac{1}{m_1} \end{pmatrix} \begin{pmatrix} -ev_{1,n} \\ m_1 v_{1,n} \\ m_1 v_{1,t} \\ m_1 v_{1,k} \end{pmatrix} = \begin{pmatrix} V_{1,n} \\ V_{1,t} \\ V_{1,k} \end{pmatrix}$$

Implementation

$$\begin{aligned} V_{t,n} &= -e v_{t,n} \\ V_{t,t} &= v_{t,t} - \mu(e+1) v_{t,n} \\ V_{t,k} &= v_{t,k} - \mu(e+1) v_{t,n} \end{aligned}$$



- $\mu(e+1)v_{t,n}$ is difficult to handle
- $|V_{t,t}| \leq |v_{t,t}|$ and $\operatorname{sign}(V_{t,t}) = \operatorname{sign}(v_{t,t})$ should be guaranteed
- $V_{t,t} = \mu v_{t,t} \quad V_{t,k} = \mu v_{t,k} \quad 0 \leq \mu \leq 1$ is a useful simplification

Position Update

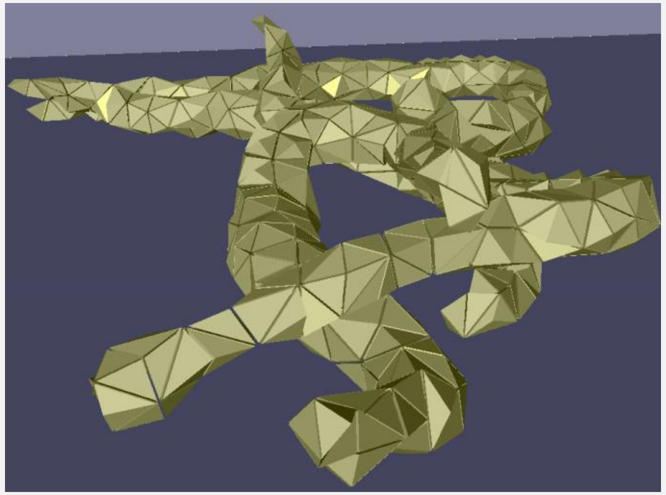
- The collision impulse updates the velocity
- However, the point is still in collision $(d < 0)$
- For low velocities, the position update in the following integration step may not be sufficient to resolve the collision
- Therefore, the position should be updated as well, e.g. $\boldsymbol{x}_{t+h} = \boldsymbol{x}^* - d \cdot \boldsymbol{n}$ which projects the point onto the plane
- The position update is not physically-motivated, it just resolves problems due to discrete time steps

Outline

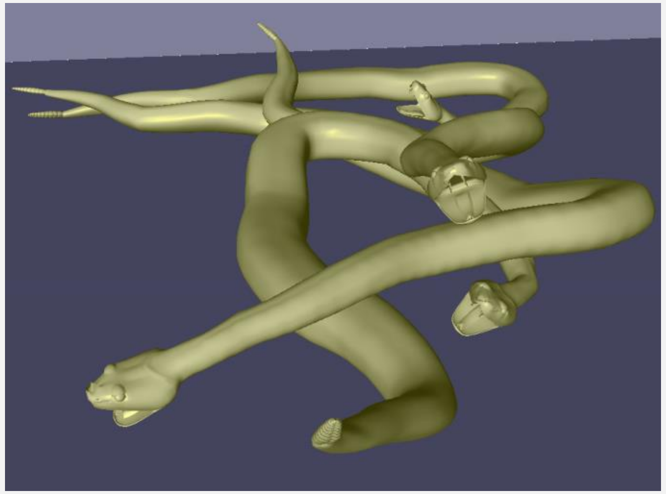
- Introduction
- Elastic forces
- Miscellaneous
- Collision handling
- Visualization

Context

- Geometric combination of
  - A low-resolution tetrahedral mesh for simulation and
  - A high-resolution triangular mesh for visualization
- Supports simplified meshing for geometrically complex surface models



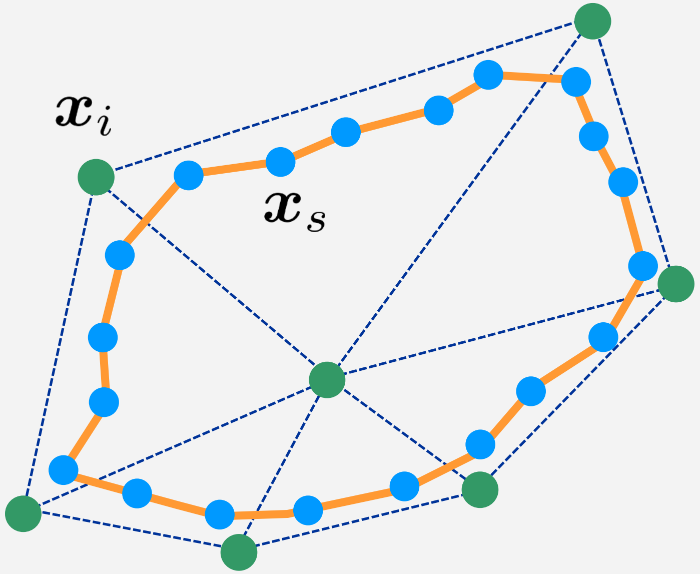
Tetrahedral mesh



Triangulated mesh

Illustration

- Two representations for simulation and visualization
- Tetrahedral elements with particles $\boldsymbol{x}_i$ are simulated
- Triangulated elements with vertices $\boldsymbol{x}_s$ are visualized



Barycentric Coordinates

- Surface vertex $\boldsymbol{x}_s$ can be represented with the particles of a tetrahedron

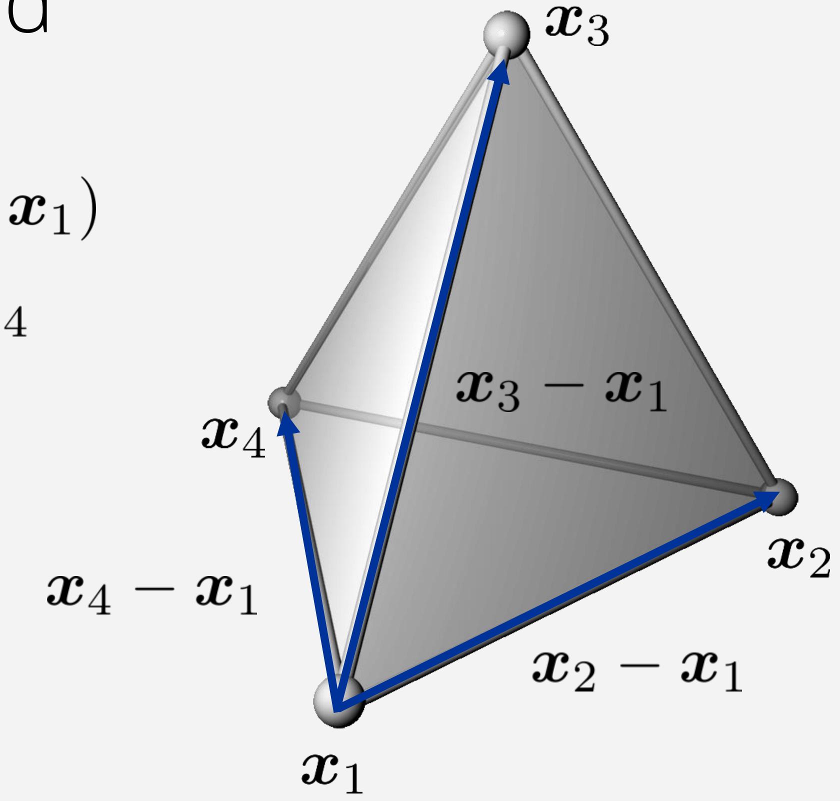
$$\boldsymbol{x}_s = \boldsymbol{x}_1 + \alpha_2(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \alpha_3(\boldsymbol{x}_3 - \boldsymbol{x}_1) + \alpha_4(\boldsymbol{x}_4 - \boldsymbol{x}_1)$$

$$\boldsymbol{x}_s = (1 - \alpha_2 - \alpha_3 - \alpha_4)\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 + \alpha_3\boldsymbol{x}_3 + \alpha_4\boldsymbol{x}_4$$

$$\boldsymbol{x}_s = \alpha_1\boldsymbol{x}_1 + \alpha_2\boldsymbol{x}_2 + \alpha_3\boldsymbol{x}_3 + \alpha_4\boldsymbol{x}_4$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$$

- $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are Barycentric coordinates of $\boldsymbol{x}_s$ with respect to $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$



Properties

- $0 < \alpha_i < 1$
  $\boldsymbol{x}_s$ is inside the convex combination of $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$,
  i.e. inside the tetrahedron
- $\alpha_i = 0 \vee \alpha_i = 1$
  $\boldsymbol{x}_s$ is on the surface of the tetrahedron
- $\alpha_i < 0 \vee \alpha_i > 1$
  $\boldsymbol{x}_s$ is outside the tetrahedron

Computation

- $\boldsymbol{x}_s = \boldsymbol{x}_1 + \alpha_2(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \alpha_3(\boldsymbol{x}_3 - \boldsymbol{x}_1) + \alpha_4(\boldsymbol{x}_4 - \boldsymbol{x}_1)$ leads to the following system

$$\left(\ (\boldsymbol{x}_2 - \boldsymbol{x}_1) \quad (\boldsymbol{x}_3 - \boldsymbol{x}_1) \quad (\boldsymbol{x}_4 - \boldsymbol{x}_1) \ \right) \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \boldsymbol{x}_s - \boldsymbol{x}_1$$

- Not solvable for degenerated tetrahedra
- α_1 is computed as $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \alpha_4$

Implementation

- Preprocessing
  - Determine the closest tetrahedron for surface points
  - Compute Barycentric coordinates for surface points with respect to the corresponding tetrahedron
- Simulation step
  - Compute surface-point positions from Barycentric coordinates and the positions of the particles of the corresponding tetrahedron
- Demo