# *Simulation in Computer Graphics*
# *Grid Fluids*

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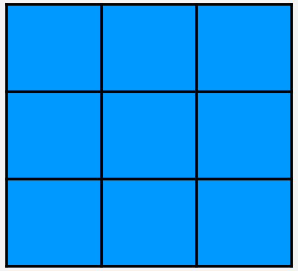
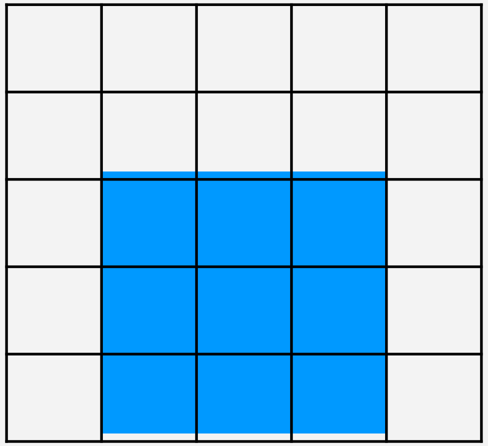
# *Motivation*



$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{g} + \nu \nabla^2 \boldsymbol{v} - \frac{1}{\rho}\nabla p$$

Lagrangian: Acceleration of a moving parcel.



$$\frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{g} + \nu \nabla^2 \boldsymbol{v} - \frac{1}{\rho}\nabla p - (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$$

Eulerian: Acceleration at a static cell.

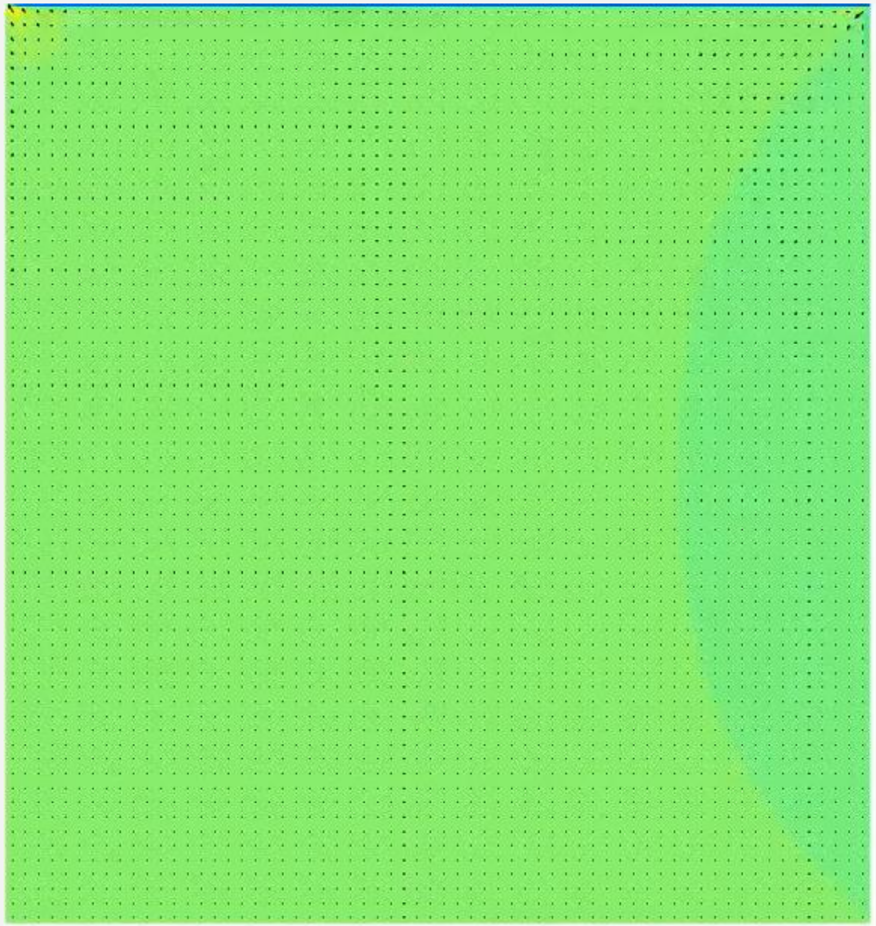
$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \boldsymbol{g} + \nu \nabla^2 \boldsymbol{v} - \frac{1}{\rho}\nabla p$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$$ or

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \quad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}$$

# Grid-based Fluids

- Benefits
  - Fixed neighbor sets
  - Constant sampling quality
  - Accuracy
  - ...
- Challenges
  - Free surfaces
  - Complex boundaries
  - Moving boundaries
  - ...



[Lorenzo Rossini]

# *Grid-based Fluids*

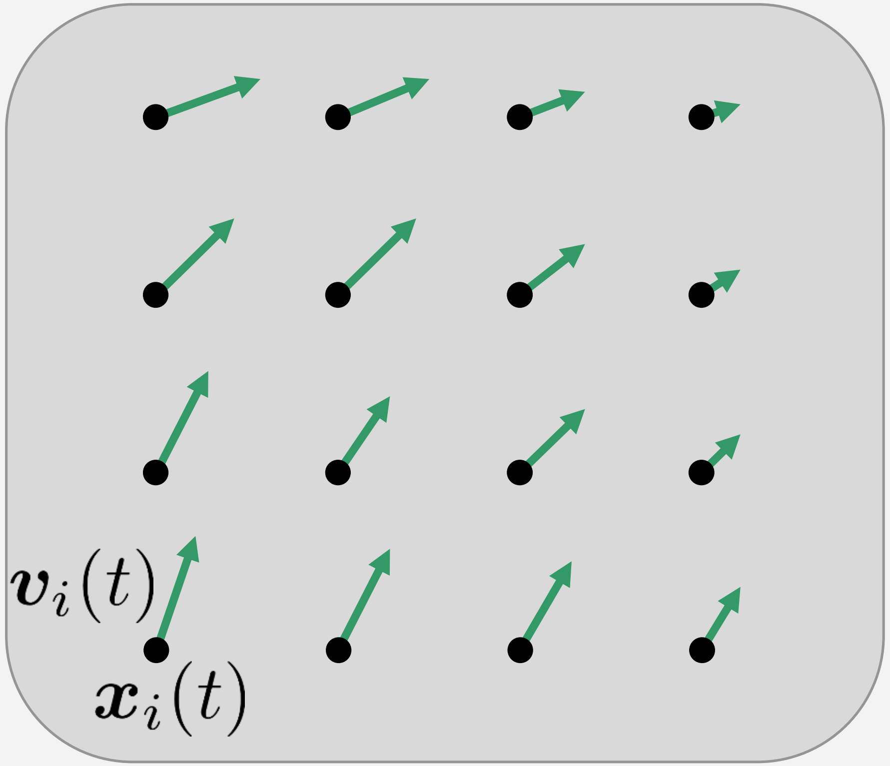
[Enright et al., SIGGRAPH 2002]

# *Outline*

- Particles vs. grids
- Advection of the velocity field
- Simple grid fluid solvers
- Discussion

# *Fluid Solvers*

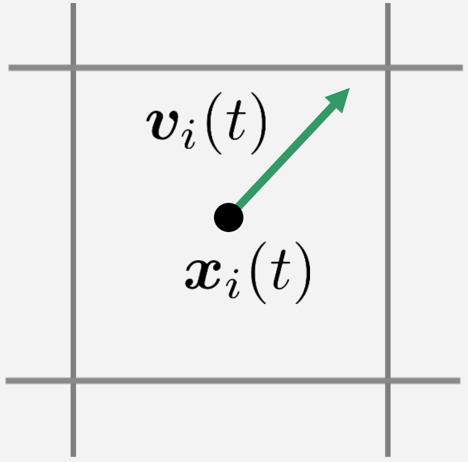
- Fluid solvers compute velocity fields that represent the fluid flow
- The velocity field is sampled at discrete time points $t$ and discrete positions $\boldsymbol{x}_i$




Velocity field at time *t*

# Particles vs. Grid Samples

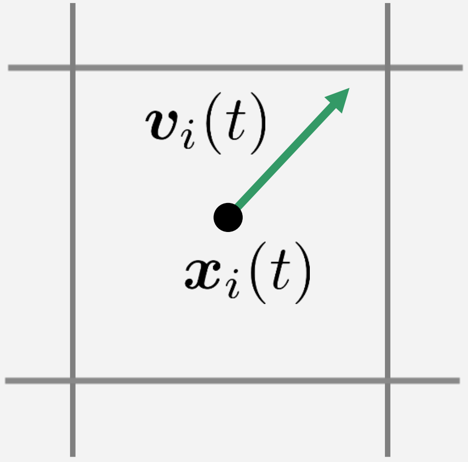
- Particles
  - Are small fractions of the fluid body
  - Represent some volume
  - Have a mass according to volume and density
  - Position moves with the flow
  - Velocity represents the velocity of the fluid parcel




$$\frac{\mathrm{d}\boldsymbol{x}_i(t)}{\mathrm{d}t} = \boldsymbol{v}_i(t)$$

# *Particles vs. Grid Samples*

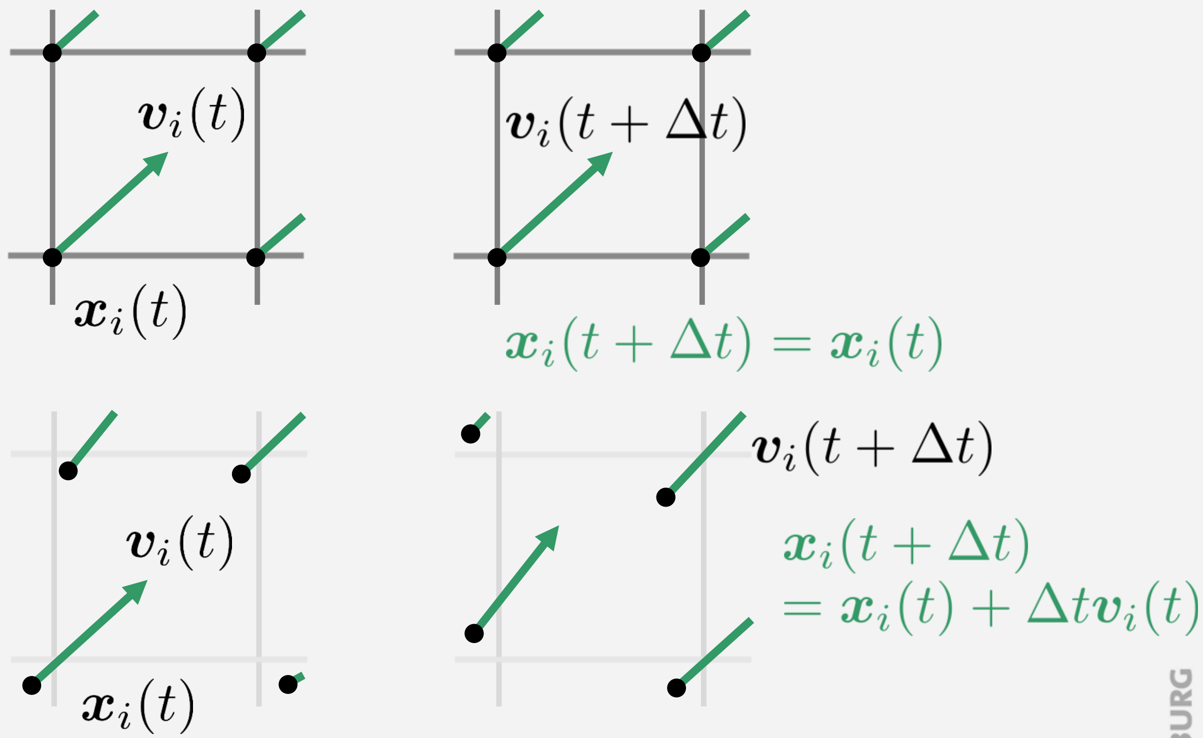
- Grid cells
  - Contain small fractions of the fluid body
  - Represent some volume
  - Contain some mass according to volume and density
  - Position is static
  - Velocity represents the flow velocity through the cell container




$$\frac{\mathrm{d}\boldsymbol{x}_i(t)}{\mathrm{d}t} = 0$$

# Particles vs. Grid Samples

- Both concepts compute the same velocity field, but typically at different sample positions
- Grid solvers do not move the samples with the flow
- Particle solvers move the samples with the flow



# Particles vs. Grid Samples

- Particles
  - Velocity updates $\boldsymbol{v}_i(t+\Delta t) = \boldsymbol{v}_i(t) + \Delta t \frac{\mathrm{d}\boldsymbol{v}_i(t)}{\mathrm{d}t}$ at a particle $i$ are computed using the Navier-Stokes equation $\frac{\mathrm{d}\boldsymbol{v}_i(t)}{\mathrm{d}t} = -\frac{1}{\rho_i(t)} \nabla p_i(t) + \nu \nabla^2 \boldsymbol{v}_i(t) + \boldsymbol{g} = \boldsymbol{a}_i(t)$
  - $\frac{\mathrm{d}\boldsymbol{v}_i(t)}{\mathrm{d}t}$ is the time rate of change of a particle, i.e. sample that is advected with the flow: $\boldsymbol{x}_i(t+\Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t)$
  - Advection of the samples accounts for the advection of the flow



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# Particles vs. Grid Samples

- Grid samples
  - Velocity updates $\boldsymbol{v}_i(t+\Delta t) = \boldsymbol{v}_i(t) + \Delta t \frac{\partial \boldsymbol{v}_i(t)}{\partial t}$ at a grid cell $i$ are computed using the Navier-Stokes equation $\frac{\partial \boldsymbol{v}_i(t)}{\partial t} = \boldsymbol{a}_i(t) - (\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)$
  - $\frac{\partial \boldsymbol{v}_i(t)}{\partial t}$ is the time rate of change of a static sample with $\boldsymbol{x}_i(t+\Delta t) = \boldsymbol{x}_i(t)$
  - $-(\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)$ accounts for the advection of the flow

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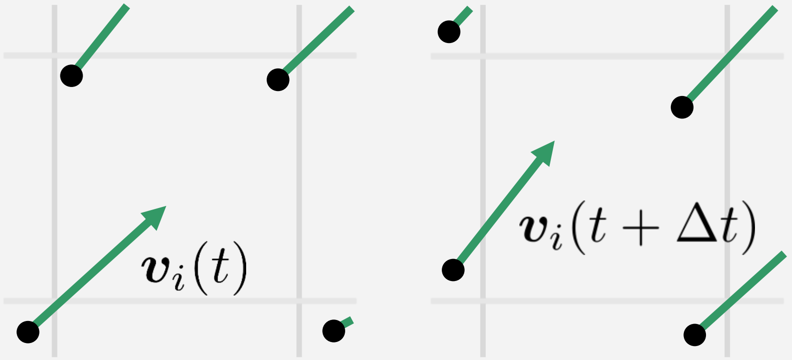
# Particles vs. Grid Samples

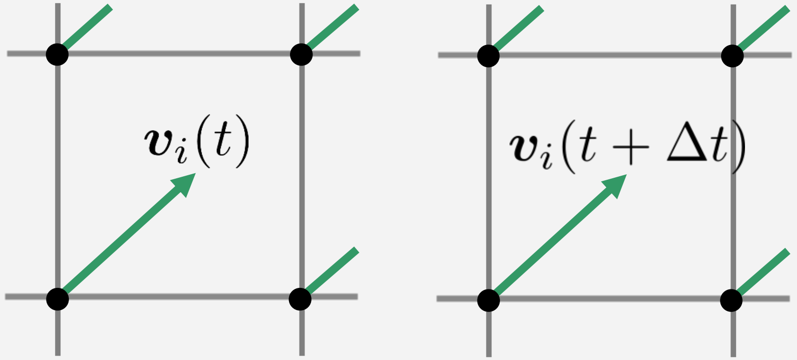
– Particles




$$\frac{\mathrm{d}\boldsymbol{v}_i(t)}{\mathrm{d}t} = \frac{\boldsymbol{v}_i(t+\Delta t) - \boldsymbol{v}_i(t)}{\Delta t}$$

$$\boldsymbol{x}_i(t+\Delta t) = \boldsymbol{x}_i(t) + \Delta t \boldsymbol{v}_i(t)$$

– Grid samples



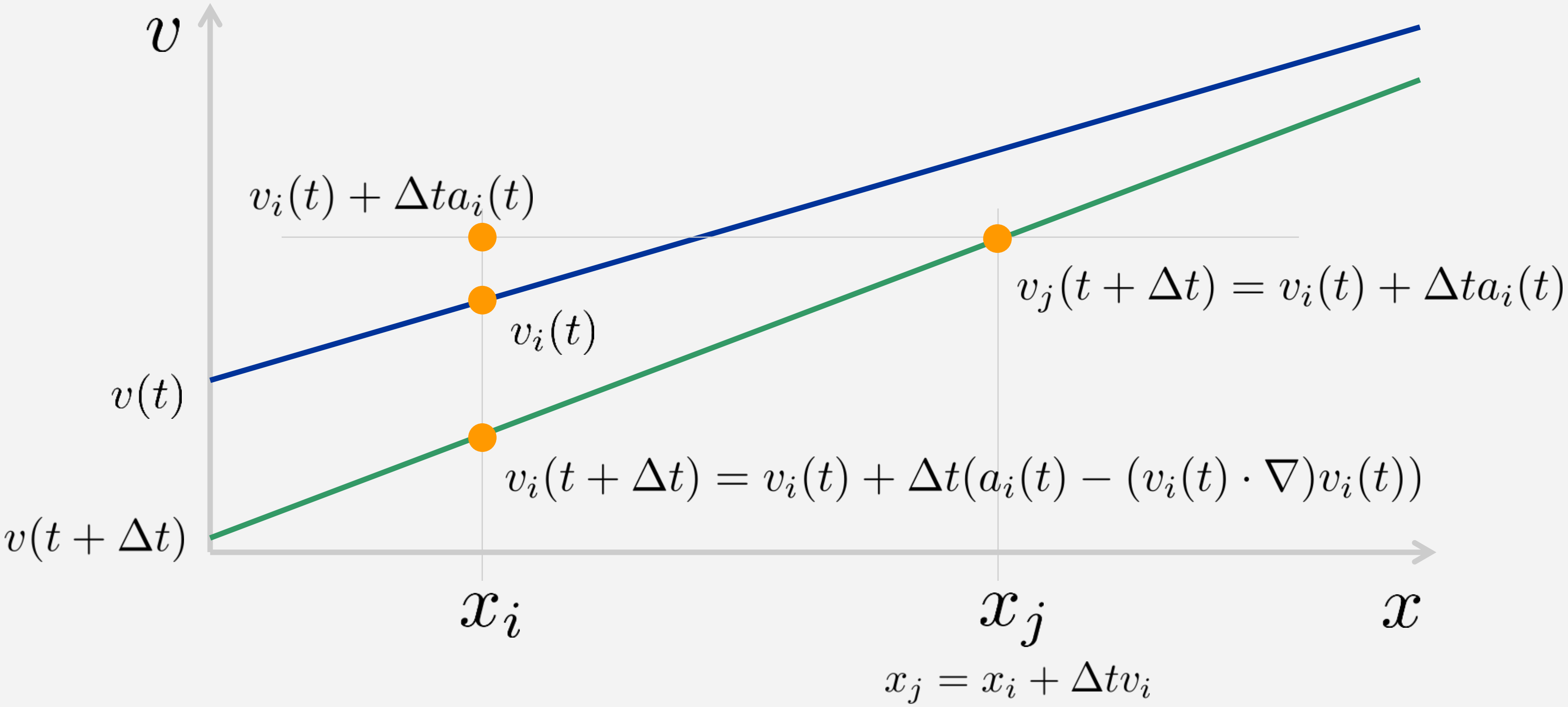

$$\frac{\partial\boldsymbol{v}_i(t)}{\partial t} = \frac{\boldsymbol{v}_i(t+\Delta t) - \boldsymbol{v}_i(t)}{\Delta t}$$

$$\boldsymbol{x}_i(t+\Delta t) = \boldsymbol{x}_i(t)$$

# Particles vs. Grids – 1D Illustration

Particle approach $v_j(t+\Delta t) = v_i(t) + \Delta t a_i(t)$

Grid approach $v_i(t+\Delta t) = v_i(t) + \Delta t(a_i(t) - (v_i(t)\cdot\nabla)v_i(t))$



# *Particles vs. Grids – 1D Illustration*

- Navier-Stokes
  - Time rate of change of the velocity of an advected position
    $$\frac{\mathrm{d}v_i(t)}{\mathrm{d}t} = a_i(t)$$
  - Time rate of change of the velocity at a fixed position
    $$\frac{\partial v_i(t)}{\partial t} = a_i(t) - (v_i(t) \cdot \nabla) v_i(t)$$



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# Particles vs. Grids – 1D Illustration

- Relation
  - Two sample positions $x_1(t_1), x_2(t_2)$ with $x_2(t_2) = x_1(t_1) + \Delta t v(x_1, t_1)$
  - Taylor approximation of the velocity

Notation

$$v(x_i, t_i) = v_i(t_i)$$

$$v(x_2, t_2) = v(x_1, t_1) + \frac{\partial v(x_1,t_1)}{\partial x}(x_2 - x_1) + \frac{\partial v(x_1,t_1)}{\partial t}(t_2 - t_1)$$

$$\frac{v(x_2,t_2) - v(x_1,t_1)}{\Delta t} = \frac{\partial v(x_1,t_1)}{\partial x}\frac{(x_2 - x_1)}{\Delta t} + \frac{\partial v(x_1,t_1)}{\partial t}\frac{(t_2 - t_1)}{\Delta t}$$

$$\frac{v(x_2,t_2) - v(x_1,t_1)}{\Delta t} = \frac{\partial v(x_1,t_1)}{\partial x} v(x_1, t_1) + \frac{\partial v(x_1,t_1)}{\partial t}$$

$$\frac{\mathrm{d}v(x_1,t_1)}{\mathrm{d}t} = \frac{\partial v(x_1,t_1)}{\partial t} + (v(x_1, t_1) \cdot \nabla) v(x_1, t_1)$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v$$

# Material Derivative

- $\frac{\mathrm{D}\boldsymbol{v}_i(t)}{\mathrm{D}t} = \frac{\mathrm{d}\boldsymbol{v}_i(t)}{\mathrm{d}t} = \frac{\partial \boldsymbol{v}_i(t)}{\partial t} + (\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)$ is the time rate of change of the velocity of a moving fluid element
- $\frac{\mathrm{D}\boldsymbol{v}_i(t)}{\mathrm{D}t} = \frac{\mathrm{d}\boldsymbol{v}_i(t)}{\mathrm{d}t}$ if $i$ is a moving particle with $\frac{\mathrm{d}\boldsymbol{x}_i(t)}{\mathrm{d}t} = \boldsymbol{v}_i(t)$
- $\frac{\mathrm{D}\boldsymbol{v}_i(t)}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}_i(t)}{\partial t} + (\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)$ if $i$ is a static grid cell
- $\frac{\mathrm{D}\boldsymbol{v}_i(t)}{\mathrm{D}t} = -\frac{1}{\rho_i(t)}\nabla p_i(t) + \nu\nabla^2\boldsymbol{v}_i(t) + \boldsymbol{g}$
  - Is a general form of the Navier-Stokes equation
  - $i$ can be a particle or a grid cell
  - Particle techniques are referred to as Lagrangian
  - Grid techniques are referred to as Eulerian

# Navier-Stokes on Grids

- $\frac{\mathrm{D}\boldsymbol{v}_i(t)}{\mathrm{D}t} = \frac{\partial \boldsymbol{v}_i(t)}{\partial t} + (\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t) = -\frac{1}{\rho_i(t)}\nabla p_i(t) + \nu \nabla^2 \boldsymbol{v}_i(t) + \boldsymbol{g}$
- Grid approaches often work with per-volume quantities in contrast to per-mass quantities
- $\rho_i(t)\left(\frac{\partial \boldsymbol{v}_i(t)}{\partial t} + (\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)\right) = -\nabla p_i(t) + \mu \nabla^2 \boldsymbol{v}_i(t) + \rho_i(t)\boldsymbol{g}$
- $\frac{\partial \boldsymbol{v}_i(t)}{\partial t} + (\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)$ is the time rate of change of the velocity of a moving fluid element
- $\frac{\partial \boldsymbol{v}_i(t)}{\partial t}$ is the local acceleration
- $(\boldsymbol{v}_i(t) \cdot \nabla)\boldsymbol{v}_i(t)$ is the convective acceleration

# Grid-based Fluid Solvers

- Grid solvers compute $\frac{\partial \boldsymbol{v}_i(t)}{\partial t}$ at all grid cells
- Velocities at grid cells are updated with, e.g.,

$$\boldsymbol{v}_i(t+\Delta t) = \boldsymbol{v}_i(t)+\Delta t \frac{\partial \boldsymbol{v}_i(t)}{\partial t} = \boldsymbol{v}_i(t)+\Delta t \left(-\frac{1}{\rho_i(t)}\nabla p_i(t) + \nu \nabla^2 \boldsymbol{v}_i(t) + \boldsymbol{g} - (\boldsymbol{v}_i(t)\cdot\nabla)\boldsymbol{v}_i(t)\right)$$

- Spatial derivatives can be approximated with, e.g., finite differences in 1D $\nabla p_i(t) = \frac{p(x_i+\Delta x,t)-p(x_i-\Delta x,t)}{2\Delta x}$ with $\Delta x$ being the grid cell size



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# *Outline*

- Particles vs. grids
- Advection of the velocity field
- Simple grid fluid solvers
- Discussion

# Overview

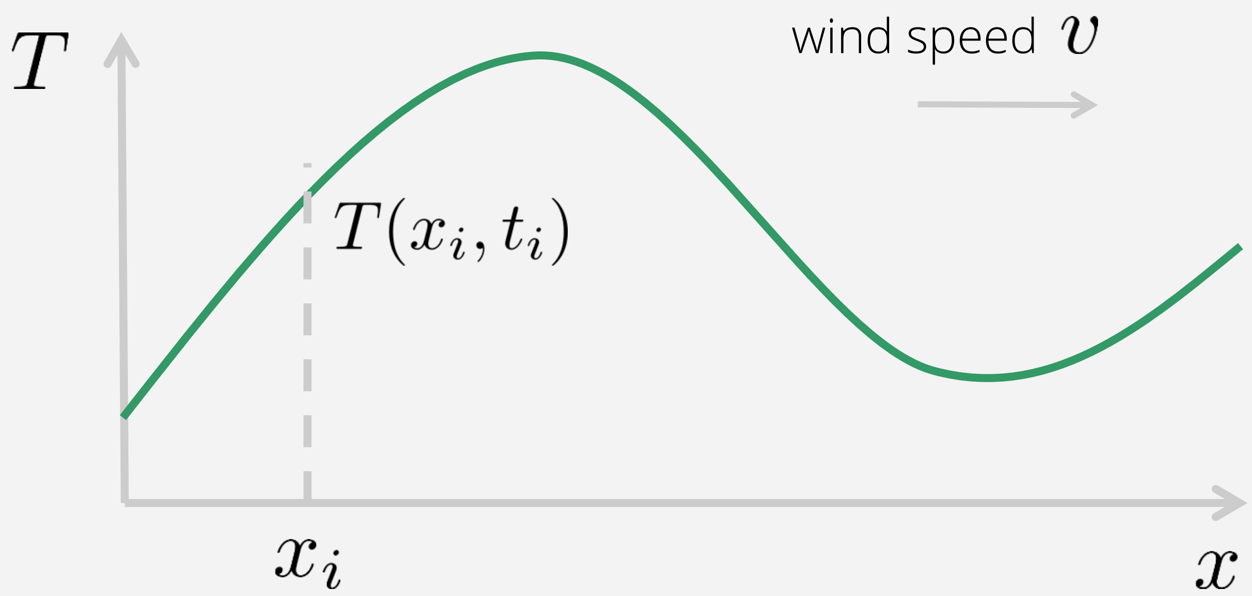
- Navier-Stokes: $\frac{\partial \boldsymbol{v}(\boldsymbol{x}_i,t)}{\partial t} = -\frac{1}{\rho(\boldsymbol{x}_i,t)}\nabla p(\boldsymbol{x}_i,t) + \nu\nabla^2\boldsymbol{v}(\boldsymbol{x}_i,t) + \boldsymbol{g} - (\boldsymbol{v}(\boldsymbol{x}_i,t)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x}_i,t)$
- Advection equation: $\frac{\partial \boldsymbol{v}(\boldsymbol{x}_i,t)}{\partial t} = -(\boldsymbol{v}(\boldsymbol{x}_i,t)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x}_i,t)$
  - Velocity $\boldsymbol{v}(\boldsymbol{x}_i,t)$ is advected by velocity $\boldsymbol{v}(\boldsymbol{x}_i,t)$
- 1D advection equation: $\frac{\partial T(x_i,t_i)}{\partial t} = -(v\cdot\nabla)T(x_i,t_i) = -v\frac{\partial T(x_i,t_i)}{\partial x}$
  - Temperature $T$ advected by constant velocity $v$
- Computation of $T(x_i,t_i+\Delta t)$ with

$$T(x_i,t_i+\Delta t) = T(x_i,t_i) + \Delta t\frac{\partial T(x_i,t_i)}{\partial t} = T(x_i,t_i) - \Delta t v\frac{\partial T(x_i,t_i)}{\partial x}$$

- Discretization of $\frac{\partial T(x_i,t_i)}{\partial x}$ with finite differences

# 1D Advection of a 1D Quantity

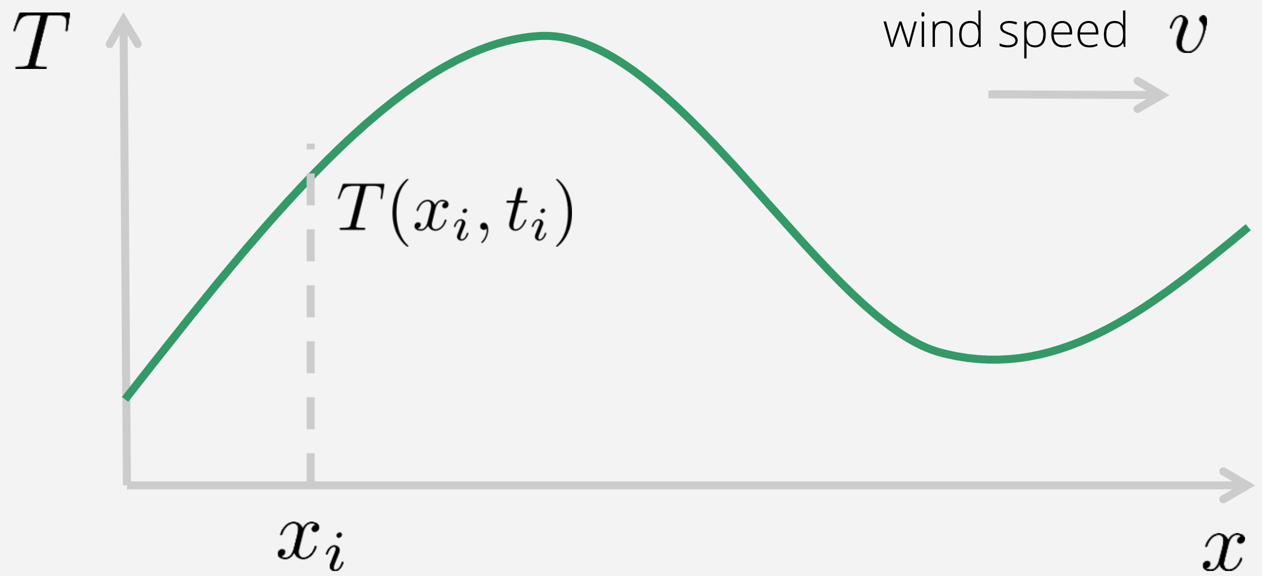
– E.g., 1D temperature field is advected by 1D velocity / wind




– $T(x_i, t_0)$ is known at all positions $x_i$ at time $t_0$

# Advection

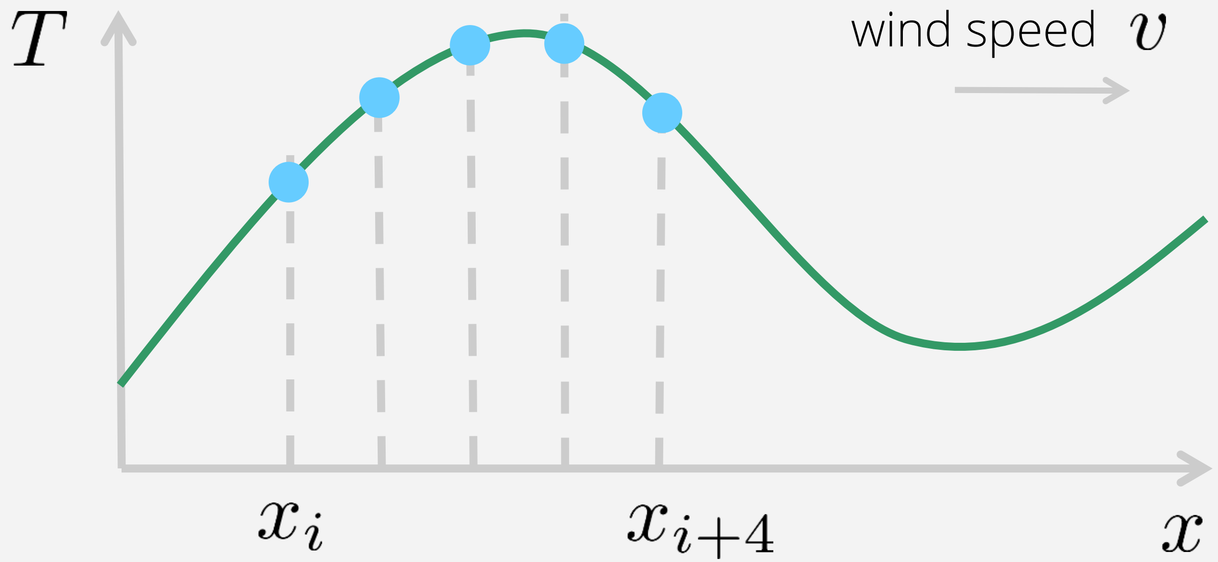
- How to compute $T(x_i, t_i)$ at arbitrary positions $x_i$ and times $t_i$?




- Analytic solution $T(x_i, t_i) = T(x_i - v \cdot (t_i - t_0), t_0)$
- $T(x_i, t_i)$ is obtained by shifting $x_i$ through a distance $v \cdot (t_i - t_0)$ without changing the shape of $T$

# Advection on a Grid – 1D

– Discretize time and space: $t_{i+1} - t_i = \Delta t \quad x_{i+1} - x_i = \Delta x$




– $T$ is considered at discrete positions and time points

# Advection on a Grid

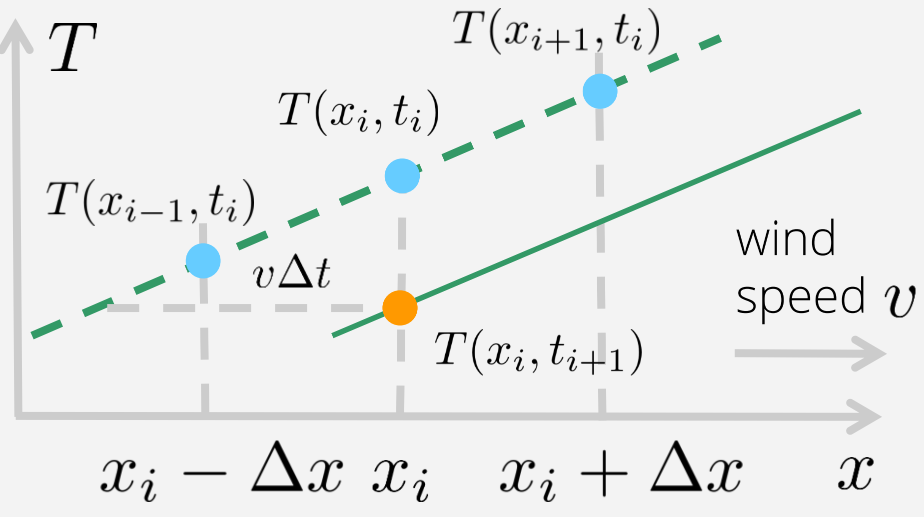
- How to compute $T(x_i, t_{i+1})$ from $T(x_i, t_i)$?
- If *T* is linear:

$$\frac{T(x_i,t_i)-T(x_i,t_{i+1})}{v\Delta t} = \frac{T(x_{i+1},t_i)-T(x_{i-1},t_i)}{2\Delta x}$$

$$\frac{T(x_i,t_{i+1})-T(x_i,t_i)}{\Delta t} = -v\frac{T(x_{i+1},t_i)-T(x_{i-1},t_i)}{2\Delta x}$$

- 1D advection equation

$$\frac{\partial T(x_i,t_i)}{\partial t} = -v\frac{\partial T(x_i,t_i)}{\partial x}$$



# 1D Advection Equation on a Grid

- The continuous form $\frac{\partial T(x_i,t_i)}{\partial t} = -v \frac{\partial T(x_i,t_i)}{\partial x}$ is discretized
- E.g., $\frac{T(x_i,t_{i+1}) - T(x_i,t_i)}{\Delta t} = -v \frac{T(x_{i+1},t_i) - T(x_{i-1},t_i)}{2\Delta x}$

  forward difference central difference

- This equation contains only one unknown $T(x_i, t_{i+1})$
- $T(x_i, t_{i+1}) = T(x_i, t_i) - \Delta t v \frac{T(x_{i+1},t_i) - T(x_{i-1},t_i)}{2\Delta x}$
- $T(x_i, t_{i+1}) = T(x_i, t_i) + \Delta t \frac{\partial T(x_i,t_i)}{\partial t} = T(x_i, t_i) - \Delta t v \frac{\partial T(x_i,t_i)}{\partial x}$
- If $T(x_i, t_i)$ is known at all samples, i.e. grid cells $x_i$, at time $t_i$, $T(x_i, t_{i+1})$ at the next time $t_{i+1}$ can be computed at all grid cells $x_i$

# *Finite Differences*

- The time derivative is commonly discretized as

$$\frac{\partial T(x_i,t_i)}{\partial t} = \frac{T(x_i,t_{i+1}) - T(x_i,t_i)}{\Delta t} + O(\Delta t)$$

- The spatial derivative is discretized in various ways

$$\frac{\partial T(x_i,t_i)}{\partial x} = \frac{T(x_{i+1},t_i) - T(x_i,t_i)}{\Delta x} + O(\Delta x)$$ forward difference

$$\frac{\partial T(x_i,t_i)}{\partial x} = \frac{T(x_i,t_i) - T(x_{i-1},t_i)}{\Delta x} + O(\Delta x)$$ backward difference

$$\frac{\partial T(x_i,t_i)}{\partial x} = \frac{T(x_{i+1},t_i) - T(x_{i-1},t_i)}{2\Delta x} + O(\Delta x^2)$$ central difference

# Solving the 1D Advection Equation

- Upwind $(v > 0)$

$$\frac{T(x_i,t_{i+1})-T(x_i,t_i)}{\Delta t} = -v\frac{T(x_i,t_i)-T(x_{i-1},t_i)}{\Delta x} \qquad T(x_i,t_{i+1}) = T(x_i,t_i) - v\Delta t\frac{T(x_i,t_i)-T(x_{i-1},t_i)}{\Delta x}$$

- Downwind $(v > 0)$

$$\frac{T(x_i,t_{i+1})-T(x_i,t_i)}{\Delta t} = -v\frac{T(x_{i+1},t_i)-T(x_i,t_i)}{\Delta x} \qquad T(x_i,t_{i+1}) = T(x_i,t_i) - v\Delta t\frac{T(x_{i+1},t_i)-T(x_i,t_i)}{\Delta x}$$

- Centered (forward time centered space FTCS)

$$\frac{T(x_i,t_{i+1})-T(x_i,t_i)}{\Delta t} = -v\frac{T(x_{i+1},t_i)-T(x_{i-1},t_i)}{2\Delta x} \qquad T(x_i,t_{i+1}) = T(x_i,t_i) - v\Delta t\frac{T(x_{i+1},t_i)-T(x_{i-1},t_i)}{2\Delta x}$$

- Leap-frog

$$\frac{T(x_i,t_{i+1})-T(x_i,t_{i-1})}{2\Delta t} = -v\frac{T(x_{i+1},t_i)-T(x_{i-1},t_i)}{2\Delta x} \qquad T(x_i,t_{i+1}) = T(x_i,t_{i-1}) - v\Delta t\frac{T(x_{i+1},t_i)-T(x_{i-1},t_i)}{\Delta x}$$

# Solving the 1D Advection Equation

- Lax-Wendroff

$$T(x_i, t_{i+1}) = T(x_i, t_i) - v\Delta t \frac{T(x_{i+1},t_i) - T(x_{i-1},t_i)}{2\Delta x} + \frac{1}{2}v^2\Delta t^2 \frac{T(x_{i+1},t_i) - 2T(x_i,t_i) + T(x_{i-1},t_i)}{\Delta x^2}$$

- Beam-Warming

$$T(x_i, t_{i+1}) = T(x_i, t_i) - v\Delta t \frac{3T(x_i,t_i) - 4T(x_{i-1},t_i) + T(x_{i-2},t_i)}{2\Delta x} + \frac{1}{2}v^2\Delta t^2 \frac{T(x_i,t_i) - 2T(x_{i-1},t_i) + T(x_{i-2},t_i)}{\Delta x^2}$$

- Lax-Friedrichs

$$T(x_i, t_{i+1}) = \frac{1}{2}(T(x_{i-1}, t_i) + T(x_{i+1}, t_i)) - v\Delta t \frac{T(x_{i+1},t_i) - T(x_{i-1},t_i)}{2\Delta x}$$

# *Finite Differences*

- Are one way to approximate spatial derivatives for grid cells
- Can be used to, e.g., compute the pressure gradient in the Navier-Stokes equation at grid cells
- Can also approximate higher-order derivatives, e.g.

$$\frac{\partial^2 T(x_i,t_i)}{\partial x^2} = \frac{T(x_{i+1},t_i) - 2T(x_i,t_i) + T(x_{i-1},t_i)}{\Delta x^2} + O(\Delta x^2)$$

- Can also compute more accurate approximations

$$\frac{\partial T(x_i,t_i)}{\partial x} = \frac{T(x_{i-2},t_i) - 8T(x_{i-1},t_i) + 8T(x_{i+1},t_i) - T(x_{i+2},t_i)}{\Delta x^2} + O(\Delta x^4)$$

$$\frac{\partial^2 T(x_i,t_i)}{\partial x^2} = \frac{-T(x_{i-2},t_i) + 16T(x_{i-1},t_i) - 30T(x_i,t_i) + 16T(x_{i+1},t_i) - T(x_{i+2},t_i)}{\Delta x^2} + O(\Delta x^4)$$

# 3D Advection Equation on a Grid

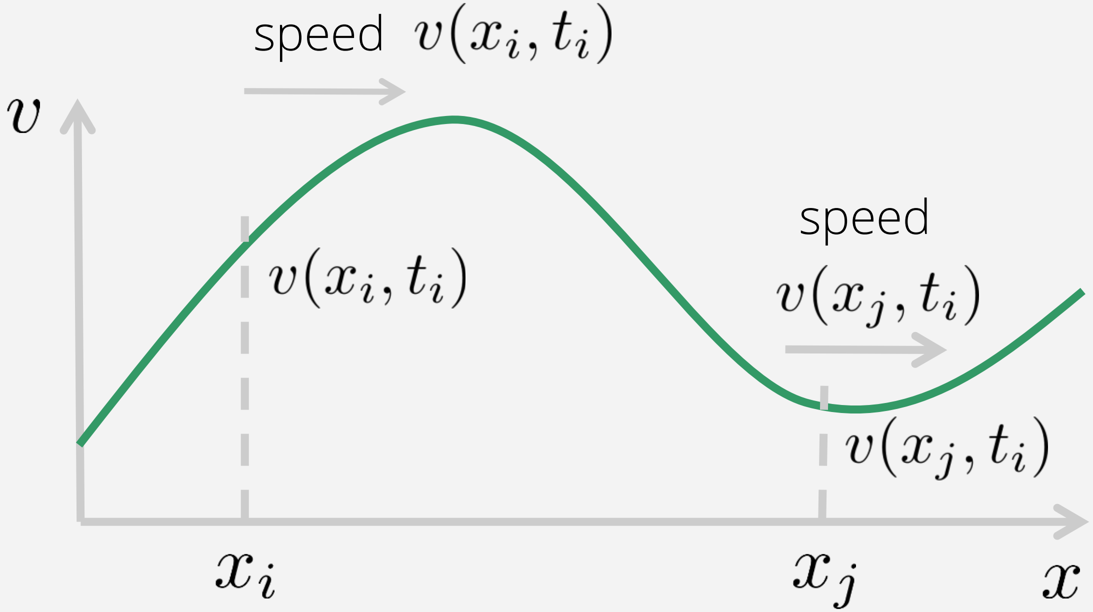
- 1D form: $\frac{\partial T(x_i,t_i)}{\partial t} = -v\frac{\partial T(x_i,t_i)}{\partial x}$
- General form: $\frac{\partial T(\boldsymbol{x}_i,t_i)}{\partial t} = -\boldsymbol{v}\nabla T(\boldsymbol{x}_i,t_i)$

grid cell position and velocity are vectors

- In 3D: $\frac{\partial T(\boldsymbol{x}_i,t_i)}{\partial t} = -\boldsymbol{v}\begin{pmatrix} \frac{\partial T(\boldsymbol{x}_i,t_i)}{\partial x_i} \\ \frac{\partial T(\boldsymbol{x}_i,t_i)}{\partial y_i} \\ \frac{\partial T(\boldsymbol{x}_i,t_i)}{\partial z_i} \end{pmatrix} = -(\boldsymbol{v}\cdot\nabla)T(\boldsymbol{x}_i,t_i)$
- Advecting a vector quantity $\boldsymbol{T}$: $\frac{\partial \boldsymbol{T}(\boldsymbol{x}_i,t_i)}{\partial t} = -(\boldsymbol{v}\cdot\nabla)\boldsymbol{T}(\boldsymbol{x}_i,t_i)$
- In the Eulerian form of the Navier-Stokes equation, the velocity at a position is advected by the velocity at that position $\frac{\partial \boldsymbol{v}(\boldsymbol{x}_i,t_i)}{\partial t} = -(\boldsymbol{v}(\boldsymbol{x}_i,t_i)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x}_i,t_i)$

# Advecting the Velocity Field

– $\frac{\partial \boldsymbol{v}(\boldsymbol{x}_i,t_i)}{\partial t} = -(\boldsymbol{v}(\boldsymbol{x}_i,t_i)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x}_i,t_i)$




– Application in the Navier-Stokes equation

$$\frac{\partial \boldsymbol{v}(\boldsymbol{x}_i,t_i)}{\partial t} = -\frac{1}{\rho(\boldsymbol{x}_i,t_i))}\nabla p(\boldsymbol{x}_i,t_i)) + \nu\nabla^2\boldsymbol{v}(\boldsymbol{x}_i,t_i) + \boldsymbol{g} - (\boldsymbol{v}(\boldsymbol{x}_i,t_i)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x}_i,t_i)$$



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# 1D/3D Velocity Advection on a Grid

– 1D: $\frac{\partial v(x_i,t_i)}{\partial t_i} = -v(x_i,t_i)\frac{\partial v(x_i,t_i)}{\partial x_i}$

– 3D: $\frac{\partial \boldsymbol{v}(\boldsymbol{x}_i,t_i)}{\partial t} = -(\boldsymbol{v}(\boldsymbol{x}_i,t_i)\cdot\nabla)\boldsymbol{v}(\boldsymbol{x}_i,t_i)$

$$= -\begin{pmatrix} v_x(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_x} & v_x(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_y} & v_x(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_z} \\ v_y(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_x} & v_y(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_y} & v_y(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_z} \\ v_z(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_x} & v_z(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_y} & v_z(\boldsymbol{x}_i,t_i)\frac{\partial}{\partial x_z} \end{pmatrix} \begin{pmatrix} v_x(\boldsymbol{x}_i,t_i) \\ v_y(\boldsymbol{x}_i,t_i) \\ v_z(\boldsymbol{x}_i,t_i) \end{pmatrix}$$

$$= -\begin{pmatrix} v_x(\boldsymbol{x}_i,t_i)\frac{\partial v_x}{\partial x_x}(\boldsymbol{x}_i,t_i) + v_y(\boldsymbol{x}_i,t_i)\frac{\partial v_x}{\partial x_y}(\boldsymbol{x}_i,t_i) + v_z(\boldsymbol{x}_i,t_i)\frac{\partial v_x}{\partial x_z}(\boldsymbol{x}_i,t_i) \\ v_x(\boldsymbol{x}_i,t_i)\frac{\partial v_y}{\partial x_x}(\boldsymbol{x}_i,t_i) + v_y(\boldsymbol{x}_i,t_i)\frac{\partial v_y}{\partial x_y}(\boldsymbol{x}_i,t_i) + v_z(\boldsymbol{x}_i,t_i)\frac{\partial v_y}{\partial x_z}(\boldsymbol{x}_i,t_i) \\ v_x(\boldsymbol{x}_i,t_i)\frac{\partial v_z}{\partial x_x}(\boldsymbol{x}_i,t_i) + v_y(\boldsymbol{x}_i,t_i)\frac{\partial v_z}{\partial x_y}(\boldsymbol{x}_i,t_i) + v_z(\boldsymbol{x}_i,t_i)\frac{\partial v_z}{\partial x_z}(\boldsymbol{x}_i,t_i) \end{pmatrix}$$

# *Outline*

- Particles vs. grids
- Advection of the velocity field
- Simple grid fluid solvers
- Discussion

# Governing Equations in 2D

- Momentum equation at position $(x_i, y_j)$ at time $t$
  - $\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{i,j}^t = -\frac{1}{\rho_{i,j}^t}\nabla p_{i,j}^t - (\boldsymbol{v}_{i,j}^t \cdot \nabla)\boldsymbol{v}_{i,j}^t$
  - Inviscid flow (artificial viscosity)
  - No body force, non-conservation form
  - State equation $p_{i,j}^t = k(\frac{\rho_{i,j}^t}{\rho_0} - 1)$
- Continuity equation at position $(x_i, y_j)$ at time $t$
  - $\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^t = -\rho_{i,j}^t \nabla \cdot \boldsymbol{v}_{i,j}^t - (\boldsymbol{v}_{i,j}^t \cdot \nabla)\rho_{i,j}^t$
  - Follows from $\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho$
  - Used for differential density update

Notation

$\boldsymbol{v}_{i,j}^t = \boldsymbol{v}(x_i, y_j, t)$



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# Governing Equations in 2D

$$\left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{i,j}^{t} = -\frac{1}{\rho_{i,j}^{t}} \nabla p_{i,j}^{t} - (\boldsymbol{v}_{i,j}^{t} \cdot \nabla)\boldsymbol{v}_{i,j}^{t}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} = -\rho_{i,j}^{t} \nabla \cdot \boldsymbol{v}_{i,j}^{t} - (\boldsymbol{v}_{i,j}^{t} \cdot \nabla)\rho_{i,j}^{t}$$

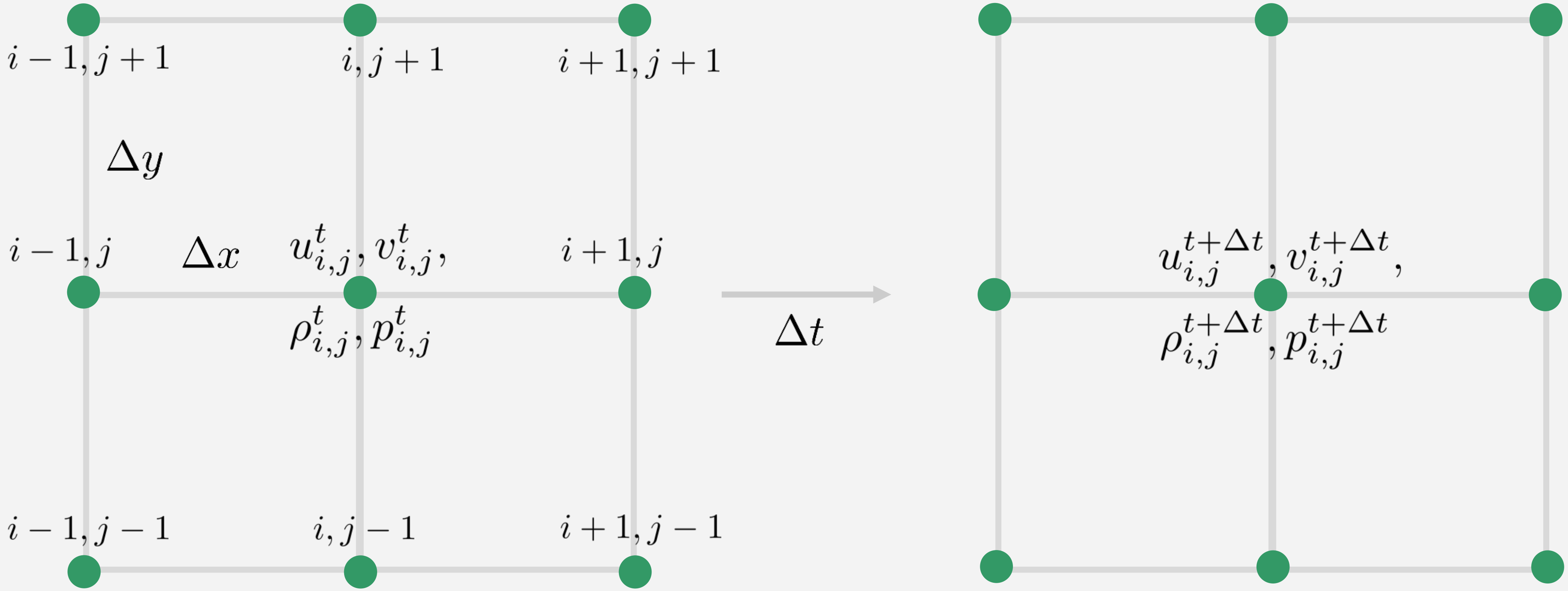
$$\boldsymbol{v}_{i,j}^{t} = (u_{i,j}^{t}, v_{i,j}^{t})^{\mathsf{T}}$$

$$\left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} = -\left(\frac{1}{\rho_{i,j}^{t}}\left(\frac{\partial p}{\partial x}\right)_{i,j}^{t} + u_{i,j}^{t}\left(\frac{\partial u}{\partial x}\right)_{i,j}^{t} + v_{i,j}^{t}\left(\frac{\partial u}{\partial y}\right)_{i,j}^{t}\right)$$

$$\left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} = -\left(\frac{1}{\rho_{i,j}^{t}}\left(\frac{\partial p}{\partial y}\right)_{i,j}^{t} + u_{i,j}^{t}\left(\frac{\partial v}{\partial x}\right)_{i,j}^{t} + v_{i,j}^{t}\left(\frac{\partial v}{\partial y}\right)_{i,j}^{t}\right)$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} = -\left(\rho_{i,j}^{t}\left(\frac{\partial u}{\partial x}\right)_{i,j}^{t} + \rho_{i,j}^{t}\left(\frac{\partial v}{\partial y}\right)_{i,j}^{t} + u_{i,j}^{t}\left(\frac{\partial \rho}{\partial x}\right)_{i,j}^{t} + v_{i,j}^{t}\left(\frac{\partial \rho}{\partial y}\right)_{i,j}^{t}\right)$$

# Solver Illustration




$$u_{i,j}^{t+\Delta t} = u_{i,j}^t + \Delta t \left(\frac{\partial u}{\partial t}\right)_{i,j}^t = u_{i,j}^t - \Delta t \left( \frac{1}{\rho_{i,j}^t} \left(\frac{\partial p}{\partial x}\right)_{i,j}^t + u_{i,j}^t \left(\frac{\partial u}{\partial x}\right)_{i,j}^t + v_{i,j}^t \left(\frac{\partial u}{\partial y}\right)_{i,j}^t \right)$$

$$= u_{i,j}^t - \Delta t \left( \frac{1}{\rho_{i,j}^t} \frac{p_{i+1,j}^t - p_{i-1,j}^t}{2\Delta x} + u_{i,j}^t \frac{u_{i+1,j}^t - u_{i-1,j}^t}{2\Delta x} + v_{i,j}^t \frac{u_{i,j+1}^t - u_{i,j-1}^t}{2\Delta y} \right)$$

# Simple 2D Grid Fluid Solver

**for all** cell $(i, j)$ **do**

$$p_{i,j}^t = k\left(\frac{\rho_{i,j}^t}{\rho_0} - 1\right)$$

Pressure from density

**for all** cell $(i, j)$ **do**

$$\left(\frac{\partial u}{\partial t}\right)_{i,j}^t = -\left(\frac{1}{\rho_{i,j}^t}\left(\frac{\partial p}{\partial x}\right)_{i,j}^t + u_{i,j}^t\left(\frac{\partial u}{\partial x}\right)_{i,j}^t + v_{i,j}^t\left(\frac{\partial u}{\partial y}\right)_{i,j}^t\right)$$

Velocity change per time

$$\left(\frac{\partial v}{\partial t}\right)_{i,j}^t = -\left(\frac{1}{\rho_{i,j}^t}\left(\frac{\partial p}{\partial y}\right)_{i,j}^t + u_{i,j}^t\left(\frac{\partial v}{\partial x}\right)_{i,j}^t + v_{i,j}^t\left(\frac{\partial v}{\partial y}\right)_{i,j}^t\right)$$

Velocity change per time

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^t = -\left(\rho_{i,j}^t\left(\frac{\partial u}{\partial x}\right)_{i,j}^t + \rho_{i,j}^t\left(\frac{\partial v}{\partial y}\right)_{i,j}^t + u_{i,j}^t\left(\frac{\partial \rho}{\partial x}\right)_{i,j}^t + v_{i,j}^t\left(\frac{\partial \rho}{\partial y}\right)_{i,j}^t\right)$$

Density change per time

**for all** cell $i, j$ **do**

$$u_{i,j}^{t+\Delta t} = u_{i,j}^t + \Delta t\left(\frac{\partial u}{\partial t}\right)_{i,j}^t$$

Velocity update for a cell

$$v_{i,j}^{t+\Delta t} = v_{i,j}^t + \Delta t\left(\frac{\partial v}{\partial t}\right)_{i,j}^t$$

Velocity update for a cell

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^t + \Delta t\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^t$$

Density update for a cell

# Simple 2D Grid Fluid Solver

- No neighbor search / fixed neighbor sets
- Pressure computation, e.g. state equation or PPE
- Spatial derivatives computed, e.g., with finite differences
  - Interestingly, SPH would also be an option
  - Pressure gradient and velocity divergence are also used in particle solvers
  - Advection terms only occur in grid solvers
- Velocity and density update per static cell
- No sample advection

# Discretization with Finite Differences

– The time derivatives

$$\left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} = -\left(\frac{1}{\rho_{i,j}^{t}}\left(\frac{\partial p}{\partial x}\right)_{i,j}^{t} + u_{i,j}^{t}\left(\frac{\partial u}{\partial x}\right)_{i,j}^{t} + v_{i,j}^{t}\left(\frac{\partial u}{\partial y}\right)_{i,j}^{t}\right)$$

$$\left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} = -\left(\frac{1}{\rho_{i,j}^{t}}\left(\frac{\partial p}{\partial y}\right)_{i,j}^{t} + u_{i,j}^{t}\left(\frac{\partial v}{\partial x}\right)_{i,j}^{t} + v_{i,j}^{t}\left(\frac{\partial v}{\partial y}\right)_{i,j}^{t}\right)$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} = -\left(\rho_{i,j}^{t}\left(\frac{\partial u}{\partial x}\right)_{i,j}^{t} + \rho_{i,j}^{t}\left(\frac{\partial v}{\partial y}\right)_{i,j}^{t} + u_{i,j}^{t}\left(\frac{\partial \rho}{\partial x}\right)_{i,j}^{t} + v_{i,j}^{t}\left(\frac{\partial \rho}{\partial y}\right)_{i,j}^{t}\right)$$

are expressed with spatial derivatives

$$\left(\frac{\partial p}{\partial x}\right)_{i,j}^{t}, \left(\frac{\partial p}{\partial y}\right)_{i,j}^{t}, \left(\frac{\partial \rho}{\partial x}\right)_{i,j}^{t}, \left(\frac{\partial \rho}{\partial y}\right)_{i,j}^{t}, \left(\frac{\partial u}{\partial x}\right)_{i,j}^{t}, \left(\frac{\partial u}{\partial y}\right)_{i,j}^{t}, \left(\frac{\partial v}{\partial x}\right)_{i,j}^{t}, \left(\frac{\partial v}{\partial y}\right)_{i,j}^{t}$$

# Discretization with Finite Differences

– E.g., using second-order central differences

$$\left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} = -\left(\frac{1}{\rho_{i,j}^{t}}\frac{p_{i+1,j}^{t}-p_{i-1,j}^{t}}{2\Delta x} + u_{i,j}^{t}\frac{u_{i+1,j}^{t}-u_{i-1,j}^{t}}{2\Delta x} + v_{i,j}^{t}\frac{u_{i,j+1}^{t}-u_{i,j-1}^{t}}{2\Delta y}\right)$$

$$\left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} = -\left(\frac{1}{\rho_{i,j}^{t}}\frac{p_{i,j+1}^{t}-p_{i,j-1}^{t}}{2\Delta y} + u_{i,j}^{t}\frac{v_{i+1,j}^{t}-v_{i-1,j}^{t}}{2\Delta x} + v_{i,j}^{t}\frac{v_{i,j+1}^{t}-v_{i,j-1}^{t}}{2\Delta y}\right)$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} = -\left(\rho_{i,j}^{t}\frac{u_{i+1,j}^{t}-u_{i-1,j}^{t}}{2\Delta x} + \rho_{i,j}^{t}\frac{v_{i,j+1}^{t}-v_{i,j-1}^{t}}{2\Delta y}\right.$$

$$\left. + u_{i,j}^{t}\frac{\rho_{i+1,j}^{t}-\rho_{i-1,j}^{t}}{2\Delta x} + v_{i,j}^{t}\frac{\rho_{i,j+1}^{t}-\rho_{i,j-1}^{t}}{2\Delta y}\right)$$

# Lax-Wendroff Technique

– Velocity and density update with

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \Delta t \left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} + \frac{\Delta t^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j}^{t}$$

$$v_{i,j}^{t+\Delta t} = v_{i,j}^{t} + \Delta t \left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} + \frac{\Delta t^2}{2}\left(\frac{\partial^2 v}{\partial t^2}\right)_{i,j}^{t}$$

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \Delta t \left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)_{i,j}^{t}$$

# Lax-Wendroff Technique - Density

- Second time derivative $\frac{\partial^2 \rho}{\partial t^2}$ can be obtained by differentiating $\frac{\partial \rho}{\partial t} = -\left(\rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y}\right)$ with respect to time

$$\frac{\partial^2 \rho}{\partial t^2} = -\left(\frac{\partial \rho}{\partial t}\frac{\partial u}{\partial x} + \rho \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial \rho}{\partial t}\frac{\partial v}{\partial y} + \rho \frac{\partial^2 v}{\partial y \partial t} + \frac{\partial u}{\partial t}\frac{\partial \rho}{\partial x} + u \frac{\partial^2 \rho}{\partial x \partial t} + \frac{\partial v}{\partial t}\frac{\partial \rho}{\partial y} + v \frac{\partial^2 \rho}{\partial y \partial t}\right)$$

- First time derivatives are computed with spatial derivatives from the governing equations

# Lax-Wendroff Technique - Density

- Mixed time / spatial derivatives, e.g. $\frac{\partial^2 u}{\partial x \partial t}$, can be obtained by differentiating, e.g. $\frac{\partial u}{\partial t} = -\left(\frac{1}{\rho}\frac{\partial p}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$ with respect to $x$

$$\frac{\partial^2 u}{\partial x \partial t} = -\left(-\frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y}\right)$$

- Discretizations of higher-order derivatives, e.g.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j}^t = \frac{\frac{u_{i+1,j}^t - u_{i,j}^t}{\Delta x} - \frac{u_{i,j}^t - u_{i-1,j}^t}{\Delta x}}{\Delta x} = \frac{u_{i+1,j}^t - 2u_{i,j}^t + u_{i-1,j}^t}{\Delta x^2}$$

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j}^t = \frac{\frac{u_{i+1,j+1}^t - u_{i-1,j+1}^t}{2\Delta x} - \frac{u_{i+1,j-1}^t - u_{i-1,j-1}^t}{2\Delta x}}{2\Delta y}$$

$$= \frac{u_{i+1,j+1}^t - u_{i-1,j+1}^t - u_{i+1,j-1}^t + u_{i-1,j-1}^t}{4\Delta x \Delta y}$$

# MacCormack Technique - Update

– Velocity and density update with

$$u_{i,j}^{t+\Delta t} = u_{i,j}^{t} + \frac{\Delta t}{2}\left[\left(\frac{\partial u}{\partial t}\right)_{i,j}^{t} + \left(\overline{\frac{\partial u}{\partial t}}\right)_{i,j}^{t+\Delta t}\right]$$

$$v_{i,j}^{t+\Delta t} = v_{i,j}^{t} + \frac{\Delta t}{2}\left[\left(\frac{\partial v}{\partial t}\right)_{i,j}^{t} + \left(\overline{\frac{\partial v}{\partial t}}\right)_{i,j}^{t+\Delta t}\right]$$

$$\rho_{i,j}^{t+\Delta t} = \rho_{i,j}^{t} + \frac{\Delta t}{2}\left[\left(\frac{\partial \rho}{\partial t}\right)_{i,j}^{t} + \left(\overline{\frac{\partial \rho}{\partial t}}\right)_{i,j}^{t+\Delta t}\right]$$

– $\left(\overline{\frac{\partial}{\partial t}}\right)_{i,j}^{t+\Delta t}$ are predicted derivatives at $t+\Delta t$ using predicted values $\overline{\rho}_{i,j}^{t+\Delta t}, \overline{p}_{i,j}^{t+\Delta t}, \overline{u}_{i,j}^{t+\Delta t}, \overline{v}_{i,j}^{t+\Delta t}$ at $t+\Delta t$

# MacCormack Technique - Prediction

- Prediction

$$\overline{u}_{i,j}^{t+\Delta t} = u_{i,j}^t + \Delta t \left(\frac{\partial u}{\partial t}\right)_{i,j}^t$$
$$= u_{i,j}^t - \Delta t \left( \frac{1}{\rho_{i,j}^t} \frac{p_{i+1,j}^t - p_{i-1,j}^t}{2\Delta x} + u_{i,j}^t \frac{u_{i+1,j}^t - u_{i-1,j}^t}{2\Delta x} + v_{i,j}^t \frac{u_{i,j+1}^t - u_{i,j-1}^t}{2\Delta y} \right)$$
$$\overline{v}_{i,j}^{t+\Delta t} = v_{i,j}^t + \Delta t \left(\frac{\partial v}{\partial t}\right)_{i,j}^t$$
$$= v_{i,j}^t - \Delta t \left( \frac{1}{\rho_{i,j}^t} \frac{p_{i,j+1}^t - p_{i,j-1}^t}{2\Delta y} + u_{i,j}^t \frac{v_{i+1,j}^t - v_{i-1,j}^t}{2\Delta x} + v_{i,j}^t \frac{v_{i,j+1}^t - v_{i,j-1}^t}{2\Delta y} \right)$$
$$\overline{\rho}_{i,j}^{t+\Delta t} = \rho_{i,j}^t + \Delta t \left(\frac{\partial \rho}{\partial t}\right)_{i,j}^t$$
$$= \rho_{i,j}^t - \Delta t \left( \rho_{i,j}^t \frac{u_{i+1,j}^t - u_{i-1,j}^t}{2\Delta x} + \rho_{i,j}^t \frac{v_{i,j+1}^t - v_{i,j-1}^t}{2\Delta y} + u_{i,j}^t \frac{\rho_{i+1,j}^t - \rho_{i-1,j}^t}{2\Delta x} + v_{i,j}^t \frac{\rho_{i,j+1}^t - \rho_{i,j-1}^t}{2\Delta y} \right)$$
$$\overline{p}_{i,j}^{t+\Delta t} = k\left(\frac{\overline{\rho}_{i,j}^{t+\Delta t}}{\rho_0} - 1\right)$$

# MacCormack Technique – Prediction

– Prediction of derivatives

$$\left(\overline{\frac{\partial u}{\partial t}}\right)_{i,j}^{t+\Delta t} = -\left(\frac{1}{\overline{\rho}_{i,j}^{t+\Delta t}}\frac{\overline{p}_{i+1,j}^{t+\Delta t}-\overline{p}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{u}_{i,j}^{t+\Delta t}\frac{\overline{u}_{i+1,j}^{t+\Delta t}-\overline{u}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{v}_{i,j}^{t+\Delta t}\frac{\overline{u}_{i,j+1}^{t+\Delta t}-\overline{u}_{i,j-1}^{t+\Delta t}}{2\Delta y}\right)$$

$$\left(\overline{\frac{\partial v}{\partial t}}\right)_{i,j}^{t+\Delta t} = -\left(\frac{1}{\overline{\rho}_{i,j}^{t+\Delta t}}\frac{\overline{p}_{i,j+1}^{t+\Delta t}-\overline{p}_{i,j-1}^{t+\Delta t}}{2\Delta y} + \overline{u}_{i,j}^{t+\Delta t}\frac{\overline{v}_{i+1,j}^{t+\Delta t}-\overline{v}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{v}_{i,j}^{t+\Delta t}\frac{\overline{v}_{i,j+1}^{t+\Delta t}-\overline{v}_{i,j-1}^{t+\Delta t}}{2\Delta y}\right)$$

$$\left(\overline{\frac{\partial \rho}{\partial t}}\right)_{i,j}^{t+\Delta t} = -\left(\overline{\rho}_{i,j}^{t+\Delta t}\frac{\overline{u}_{i+1,j}^{t+\Delta t}-\overline{u}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{\rho}_{i,j}^{t+\Delta t}\frac{\overline{v}_{i,j+1}^{t+\Delta t}-\overline{v}_{i,j-1}^{t+\Delta t}}{2\Delta y}\right.$$
$$\left. + \overline{u}_{i,j}^{t+\Delta t}\frac{\overline{\rho}_{i+1,j}^{t+\Delta t}-\overline{\rho}_{i-1,j}^{t+\Delta t}}{2\Delta x} + \overline{v}_{i,j}^{t+\Delta t}\frac{\overline{\rho}_{i,j+1}^{t+\Delta t}-\overline{\rho}_{i,j-1}^{t+\Delta t}}{2\Delta y}\right)$$

# *Outline*

– Particles vs. grids
– Advection of the velocity field
– Simple grid fluid solvers
– Discussion

# Discussion

- Boundary handling
  - Same concepts as for particle fluids
  - Boundary samples with predefined values, e.g. mirrored or extrapolated pressure
- Time step
  - Same rules as for particle fluids (CFL number)
  - Velocity times time step should be smaller than cell size
- Viscosity
  - Grid solvers typically suffer from significant artificial viscosity

# Discussion

- Staggered grid
  - Velocity and pressure considered at shifted positions
  - Requires interpolations
- Free surface
  - Level sets (initial interface is advected with the flow)
  - Tracer particles (semi-Lagrangian)
- Simulation step
  - Typically subdivided into advection $\boldsymbol{v}^*_{i,j} = \boldsymbol{v}^t_{i,j} - \Delta t(\boldsymbol{v}^t_{i,j} \cdot \nabla)\boldsymbol{v}^t_{i,j}$ followed by projection $\boldsymbol{v}^{t+\Delta t}_{i,j} = \boldsymbol{v}^*_{i,j} - \Delta t \frac{1}{\rho^t_{i,j}} \nabla p^t_{i,j}$

# Discussion – Grid Solvers in Graphics

- Advection
  - $\boldsymbol{v}^*_{i,j} = \boldsymbol{v}^t_{i,j} - \Delta t(\boldsymbol{v}^t_{i,j} \cdot \nabla)\boldsymbol{v}^t_{i,j}$
  - Typically realized with tracer particles
    - Independent particles are advected with the flow
    - Interpolation of particle velocities from / to cell velocities
    - PIC, FLIP, Stam's stable fluid

# *Discussion – Grid Solvers in Graphics*

- Projection
  - $\boldsymbol{v}_{i,j}^{t+\Delta t} = \boldsymbol{v}_{i,j}^{*} - \Delta t \frac{1}{\rho_{i,j}^{t}} \nabla p_{i,j}^{t}$
  - Pressure is typically computed with a PPE
    - Divergence of predicted velocity as source term
    - No explicit notion of density deviation

# Grid-based Fluids


[Carlson et al., SIGGRAPH 2004]