Simulation in Computer Graphics
Rigid Bodies

Matthias Teschner

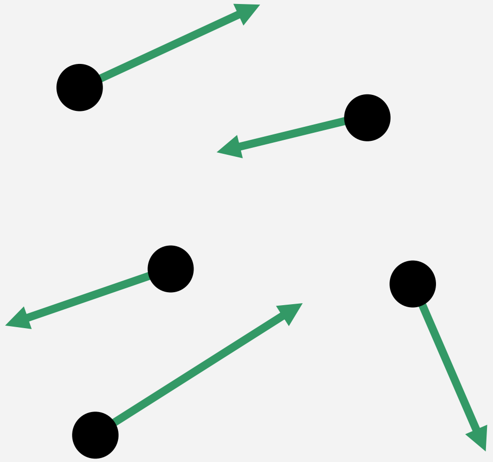
Elastic Solid vs. Rigid Body

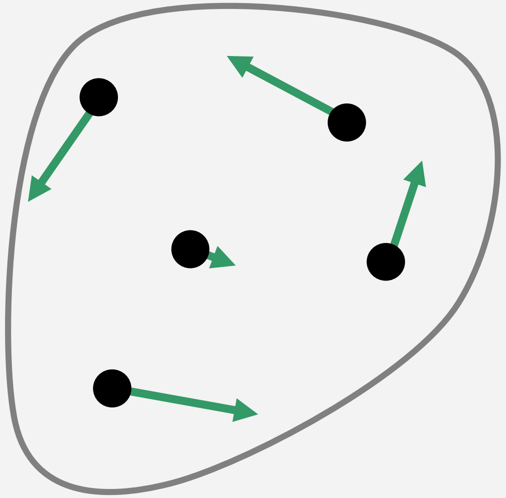
Elastic Solid

- Represented with *n* particles
- Relative movements
- Mass, position, velocity, force considered per particle
- No notion of orientation

Rigid Body

- Represented with *n* particles
- No relative movements
- Interaction implicitly modeled
- One position / one orientation
- One velocity / one angular velocity
- Mass and mass distribution

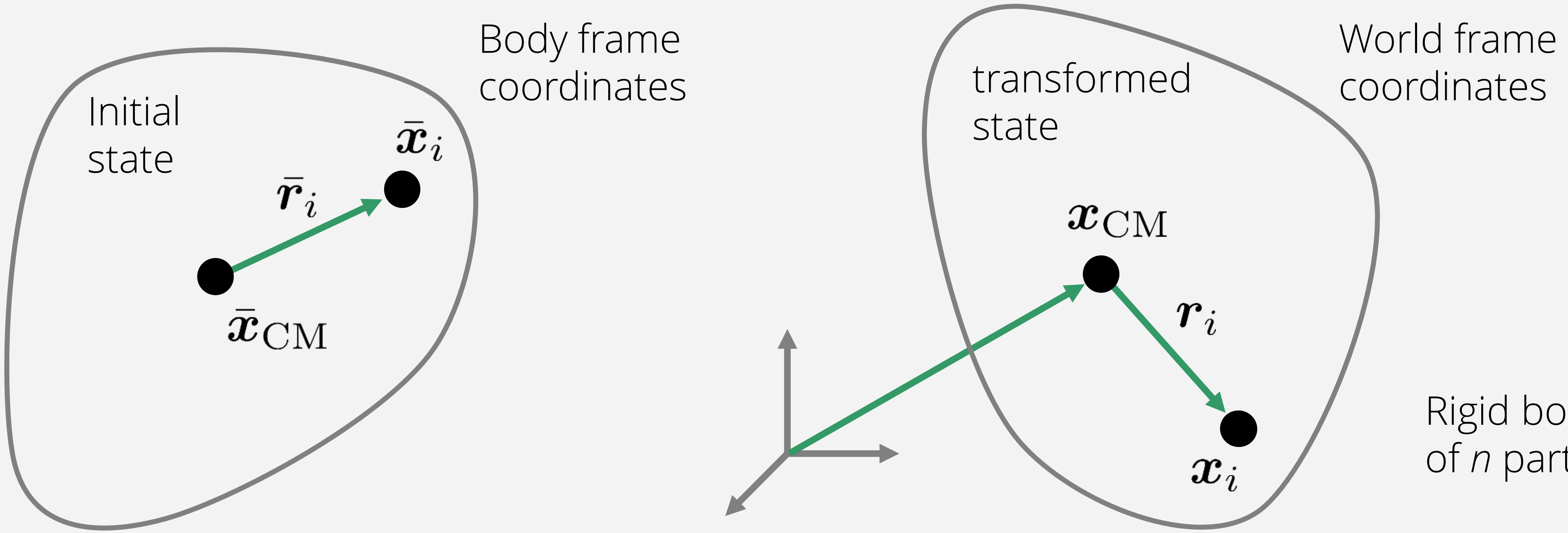




Outline

- Position and orientation
- Linear and angular velocity
- Mass and inertia tensor
- Linear and angular momentum
- Force and torque
- Simulation loop

Particle Representation



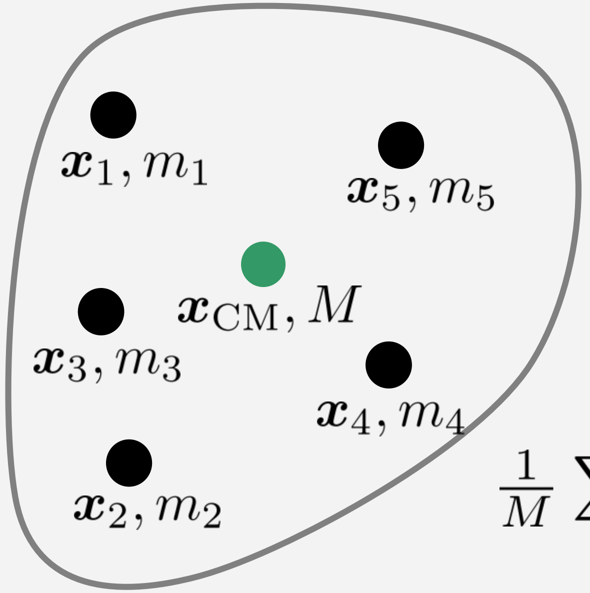
Rigid body consists of *n* particles.

$\bar{\boldsymbol{x}}_{\mathrm{CM}} = (0,0,0)^{\mathsf{T}}$ Reference position

$\bar{\boldsymbol{r}}_i = \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_{\mathrm{CM}}$ Position of a particle relative to the reference position

$\boldsymbol{x}_i = \boldsymbol{x}_{\mathrm{CM}} + \boldsymbol{r}_i = \bar{\boldsymbol{x}}_{\mathrm{CM}} + \boldsymbol{t} + \mathrm{Rot}(\bar{\boldsymbol{r}}_i)$ Particle position in global coordinates. T, Rot are translation and rotation of the rigid body

Reference Position – Center of Mass



$$\boldsymbol{x}_{\text{CM}} = \frac{\sum_i m_i \boldsymbol{x}_i}{\sum_i m_i} = \frac{\sum_i m_i \boldsymbol{x}_i}{M}$$

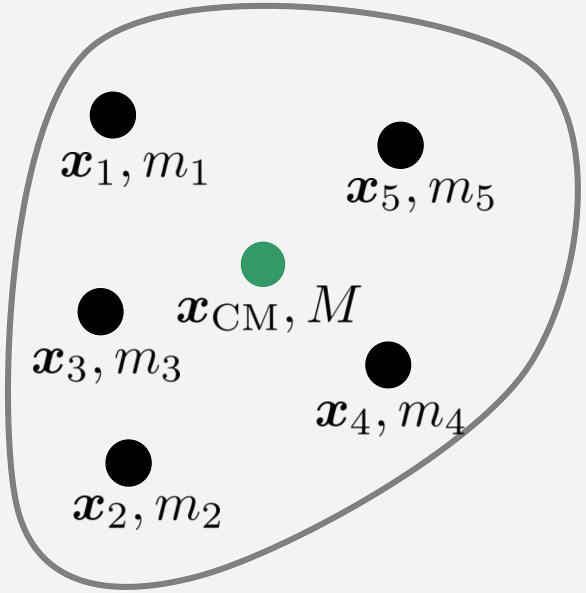
Center of mass

$$M\boldsymbol{x}_{\text{CM}} = \sum_i m_i \boldsymbol{x}_i$$

$$\frac{1}{M} \sum_i m_i \boldsymbol{T} \boldsymbol{x}_i = \boldsymbol{T} \frac{1}{M} \sum_i m_i \boldsymbol{x}_i = \boldsymbol{T} \boldsymbol{x}_{\text{CM}}$$

Same point under translation and rotation

Center of Mass - Property



$$\boldsymbol{f}_i = m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_i$$ Force at position x_i

$$\boldsymbol{F} = \sum_i \boldsymbol{f}_i = \sum_i m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_i = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_i m_i \boldsymbol{x}_i$$ Overall force

$$= \frac{\mathrm{d}^2}{\mathrm{d}t^2} M \boldsymbol{x}_{\mathrm{CM}} = M \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_{\mathrm{CM}}$$

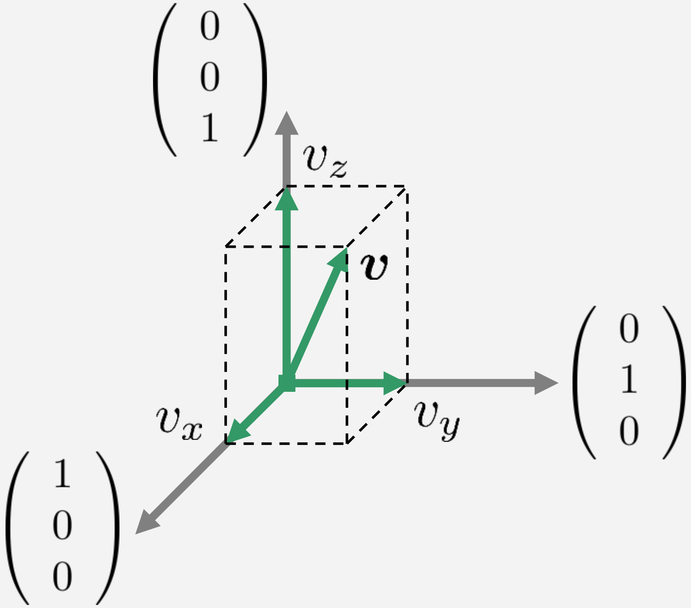
$$\boldsymbol{F} = \sum_i \boldsymbol{f}_i = M \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_{\mathrm{CM}}$$

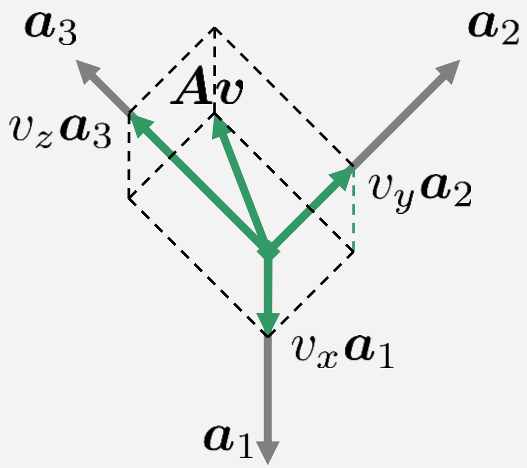
The linear acceleration of the center of mass can be computed from the mass of the rigid body and from the sum of all forces acting at arbitrary rigid body positions. The positions of the applied forces do not influence the linear acceleration.

Orientation in 3D

- Rotation matrix
  - 3x3 elements to represent 3 degrees of freedom

$$\mathrm{Rot}(\boldsymbol{v}) = \boldsymbol{A}\boldsymbol{v} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = v_x\boldsymbol{a}_1 + v_y\boldsymbol{a}_2 + v_z\boldsymbol{a}_3$$





Orientation in 3D

- $\boldsymbol{A}$ has to be orthonormal

$$\boldsymbol{A}\boldsymbol{A}^\intercal = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{a}_1^\intercal \\ \boldsymbol{a}_2^\intercal \\ \boldsymbol{a}_3^\intercal \end{pmatrix} = \boldsymbol{I}$$

$$\mathrm{Det}(\boldsymbol{A}) = 1$$

- Magnitude of eigenvalues is 1, so $\boldsymbol{A}$ preserves the length of $\boldsymbol{r}$
- Position of a body point $\boldsymbol{x}_i = \boldsymbol{x}_{\mathrm{CM}} + \boldsymbol{r}_i = \bar{\boldsymbol{x}}_{\mathrm{CM}} + \boldsymbol{t} + \boldsymbol{A}\bar{\boldsymbol{r}}_i$

translation + rotation

Body in Motion

- Time-dependent position

$$\boldsymbol{x}_i(t) = \bar{\boldsymbol{x}}_{\mathrm{CM}} + \boldsymbol{t}(t) + \boldsymbol{A}(t)\bar{\boldsymbol{r}}_i = \boldsymbol{x}_{\mathrm{CM}}(t) + \boldsymbol{A}(t)\bar{\boldsymbol{r}}_i$$

- Velocity

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_{\mathrm{CM}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}(t)\bar{\boldsymbol{r}}_i + \boldsymbol{A}(t)\frac{\mathrm{d}}{\mathrm{d}t}\bar{\boldsymbol{r}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_{\mathrm{CM}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}(t)\bar{\boldsymbol{r}}_i$$

linear velocity + angular velocity

Angular Velocity in 3D

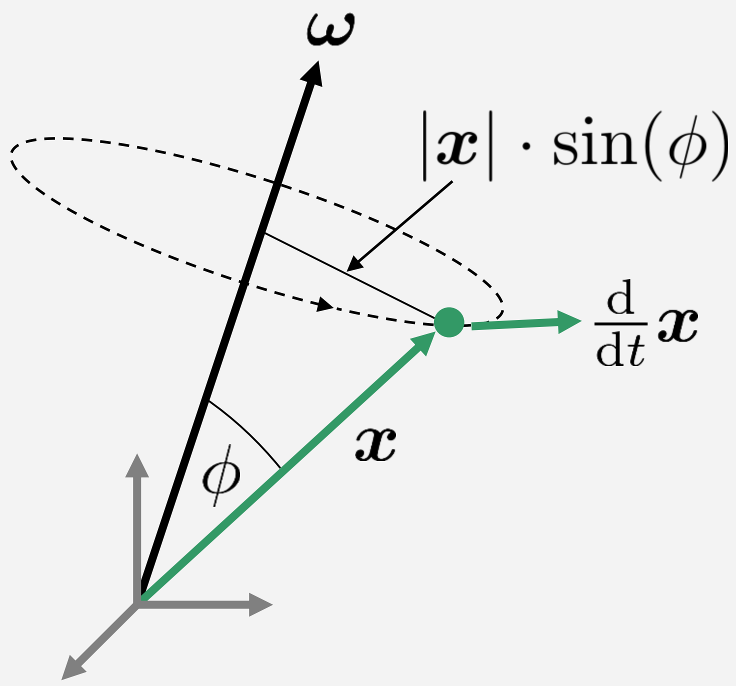
- Angular velocity $\boldsymbol{\omega}$ is a 3D vector
  - In direction of axis of rotation
  - $|\boldsymbol{\omega}|$ is the magnitude of the angular velocity [rad/s]

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}\right| = |\boldsymbol{\omega}| \cdot r = |\boldsymbol{\omega}| \cdot |\boldsymbol{x}| \cdot \sin(\phi)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}$$

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \rightarrow \tilde{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \tilde{\omega}\mathbf{x}$$



Rigid Body Kinematics

– What is the relation between $\tilde{\boldsymbol{\omega}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}$?

– Angular velocity rotates all axis (columns of $\boldsymbol{A}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}_1 & \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}_2 & \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{a}_1 & \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{a}_2 & \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{a}_3 \end{pmatrix} = \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{A}$$

– Velocity of a point

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_i(t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_{\mathrm{CM}}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}(t) \cdot \bar{\boldsymbol{r}}_i$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_i(t) = \boldsymbol{v}(t) + \tilde{\boldsymbol{\omega}}(t) \cdot \boldsymbol{A}(t) \cdot \bar{\boldsymbol{r}}_i$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_i(t) = \boldsymbol{v}(t) + \tilde{\boldsymbol{\omega}}(t) \cdot \left(\boldsymbol{x}_i(t) - \boldsymbol{x}_{\mathrm{CM}}(t)\right)$$

State Vector

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{v}(t) \end{pmatrix}$$

Particle

$$\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{A}(t) \\ \boldsymbol{v}(t) \\ \boldsymbol{\omega}(t) \end{pmatrix}$$

Rigid body

Rigid Body Dynamics

- Forces change
  - Linear velocity
  - Angular velocity
- Linear velocity change

$$\boldsymbol{F} = \sum_i \boldsymbol{f}_i = \sum_i m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_i = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_i m_i \boldsymbol{x}_i$$

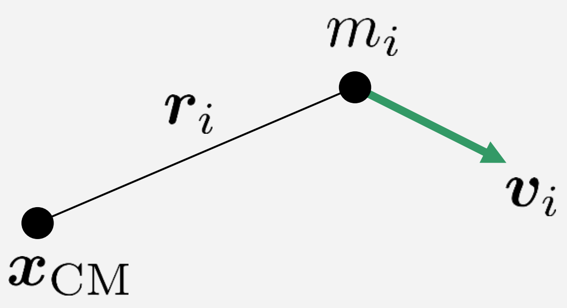
$$= \frac{\mathrm{d}^2}{\mathrm{d}t^2} M \boldsymbol{x}_{\mathrm{CM}} = M \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_{\mathrm{CM}}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{x}_{\mathrm{CM}} = \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{v} = \frac{\boldsymbol{F}}{M} = \frac{\sum_i \boldsymbol{f}_i}{M}$$

- Like a particle, but …

Angular Momentum

– The angular momentum of a particle w.r.t. the center of mass is



$$\boldsymbol{L}_i = \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \boldsymbol{r}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{r}_i)$$

– The total angular momentum of the body is

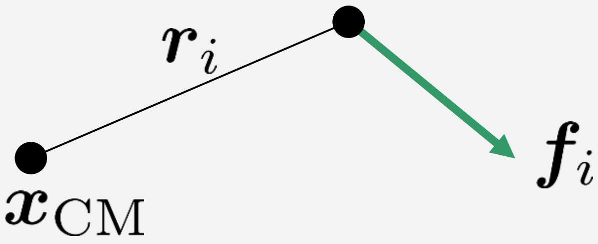
$$\begin{aligned}\boldsymbol{L} &= \sum_i \boldsymbol{L}_i = \sum_i \boldsymbol{r}_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{r}_i) \\ &= \sum_i -m_i \tilde{\boldsymbol{r}}_i \tilde{\boldsymbol{r}}_i \boldsymbol{\omega} = \left(\sum_i -m_i \tilde{\boldsymbol{r}}_i \tilde{\boldsymbol{r}}_i\right) \cdot \boldsymbol{\omega} \\ &= \boldsymbol{I}\boldsymbol{\omega}\end{aligned}$$

Inertia Tensor

- The total angular momentum is $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$ with $\boldsymbol{I}$ being a 3x3 matrix (the inertia tensor of the body)
- $\boldsymbol{I}$ depends on the rotated configuration $\boldsymbol{I} = \sum_i -m_i \tilde{\boldsymbol{r}}_i \tilde{\boldsymbol{r}}_i$
- The inertia tensor for the original body can be pre-computed, e.g. $\bar{\boldsymbol{I}} = \sum_i -m_i \tilde{\bar{\boldsymbol{r}}}_i \tilde{\bar{\boldsymbol{r}}}_i$
- Similarity transform relates time-dependent $\boldsymbol{I}$ and pre-computed $\bar{\boldsymbol{I}}$: $\boldsymbol{I} = \boldsymbol{A}\bar{\boldsymbol{I}}\boldsymbol{A}^\mathsf{T}$

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega} = \boldsymbol{A}\bar{\boldsymbol{I}}\boldsymbol{A}^\mathsf{T}\boldsymbol{A}\bar{\omega} = \boldsymbol{A}\bar{\boldsymbol{I}}\bar{\boldsymbol{\omega}}$$

Torque



- The torque of a particle w.r.t. the center of mass is

  $\boldsymbol{\tau}_i = \boldsymbol{r}_i \times \boldsymbol{f}_i$

- The total torque of the body is

  $\boldsymbol{\tau} = \sum_i \boldsymbol{\tau}_i = \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i$

Newton's Second Law (Angular)

- Angular momentum $\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times m_i \boldsymbol{v}_i = \boldsymbol{I}\boldsymbol{\omega}$
- Torque $\boldsymbol{\tau} = \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i$
- The angular version of Newton's Second law reads
  $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{\tau}$
- Tells us, how the forces $\boldsymbol{f}_i$
  change the angular velocity $\boldsymbol{\omega}$
  $\boldsymbol{\tau} = \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i$
  $\boldsymbol{L} = \boldsymbol{L} + \Delta t \cdot \boldsymbol{\tau}$
  $\boldsymbol{\omega} = \boldsymbol{I}^{-1}\boldsymbol{L}$

Linear vs. Angular Quantities

Linear momentum

$$\boldsymbol{p} = M\boldsymbol{v}$$

Linear velocity

$$\boldsymbol{v} = M^{-1}\boldsymbol{p}$$

Time-derivative of the linear momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p} = \boldsymbol{F}$$

Angular momentum

$$\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\omega}$$

Angular velocity

$$\boldsymbol{\omega} = \boldsymbol{I}^{-1}\boldsymbol{L}$$

Time-derivative of the angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \boldsymbol{\tau}$$

Governing Equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \boldsymbol{x}(t) \\ \boldsymbol{A}(t) \\ \boldsymbol{p}(t) \\ \boldsymbol{L}(t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}(t) \\ \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{A}(t) \\ \boldsymbol{F}(t) \\ \boldsymbol{\tau}(t) \end{pmatrix}$$

Simulation Step

Pre-computation

$$M \leftarrow \sum_i m_i$$

$$\bar{\boldsymbol{x}}_{\text{CM}} \leftarrow \frac{1}{M} \sum_i \bar{\boldsymbol{x}}_i m_i$$

$$\bar{\boldsymbol{r}}_i \leftarrow \bar{\boldsymbol{x}}_i - \bar{\boldsymbol{x}}_{\text{CM}}$$

$$\bar{\boldsymbol{I}}^{-1} \leftarrow \left(\sum_i -m_i \tilde{\bar{\boldsymbol{r}}}_i \tilde{\bar{\boldsymbol{r}}}_i\right)^{-1}$$

Initialization

$$\boldsymbol{x}_{\text{CM}}, \boldsymbol{v}_{\text{CM}}, \boldsymbol{A}, \boldsymbol{L}$$

$$\boldsymbol{I}^{-1} \leftarrow \boldsymbol{A} \bar{\boldsymbol{I}}^{-1} \boldsymbol{A}^{\text{T}}$$

$$\boldsymbol{\omega} \leftarrow \boldsymbol{I}^{-1} \boldsymbol{L}$$

Simulation step

Per-particle quantities

$$\boldsymbol{\tau} \leftarrow \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i$$

$$\boldsymbol{F} \leftarrow \sum_i \boldsymbol{f}_i$$

Per-body quantities

$$\boldsymbol{x}_{\text{CM}} \leftarrow \boldsymbol{x}_{\text{CM}} + \Delta t \cdot \boldsymbol{v}_{\text{CM}}$$

$$\boldsymbol{v}_{\text{CM}} \leftarrow \boldsymbol{v}_{\text{CM}} + \Delta t \cdot \boldsymbol{F}/M$$

$$\boldsymbol{A} \leftarrow \boldsymbol{A} + \Delta t \cdot \tilde{\boldsymbol{\omega}} \boldsymbol{A}$$

$$\boldsymbol{L} \leftarrow \boldsymbol{L} + \Delta t \cdot \boldsymbol{\tau}$$

$$\boldsymbol{I}^{-1} \leftarrow \boldsymbol{A} \bar{\boldsymbol{I}}^{-1} \boldsymbol{A}^{\text{T}}$$

$$\boldsymbol{\omega} \leftarrow \boldsymbol{I}^{-1} \boldsymbol{L}$$

Per-particle quantities

$$\boldsymbol{r}_i \leftarrow \boldsymbol{A} \cdot \bar{\boldsymbol{r}}_i$$

$$\boldsymbol{x}_i \leftarrow \boldsymbol{x}_{\text{CM}} + \boldsymbol{r}_i$$

$$\boldsymbol{v}_i \leftarrow \boldsymbol{v}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{r}_i$$

Reorthonormalization of the Orientation

- Orientation matrix is updated with $\boldsymbol{A} \leftarrow \boldsymbol{A} + \Delta t \cdot \tilde{\boldsymbol{\omega}} \boldsymbol{A}$
- Errors accumulate
- $\boldsymbol{A}$ is not orthonormal anymore
- Gram-Schmidt orthonormalization

$$\boldsymbol{b}_1 = \boldsymbol{a}_1 / |\boldsymbol{a}_1|$$
$$\boldsymbol{b}_2 = \boldsymbol{a}_2 - (\boldsymbol{b}_1 \cdot \boldsymbol{a}_2)\boldsymbol{b}_1$$
$$\boldsymbol{b}_2 = \boldsymbol{b}_2 / |\boldsymbol{b}_2|$$
$$\boldsymbol{b}_3 = \boldsymbol{a}_3 - (\boldsymbol{b}_1 \cdot \boldsymbol{a}_3)\boldsymbol{b}_1 - (\boldsymbol{b}_2 \cdot \boldsymbol{a}_3)\boldsymbol{b}_2$$
$$\boldsymbol{b}_3 = \boldsymbol{b}_3 / |\boldsymbol{b}_3|$$