Simulation in Computer Graphics

Exercises - Notes

Matthias Teschner

Computer Science Department
University of Freiburg

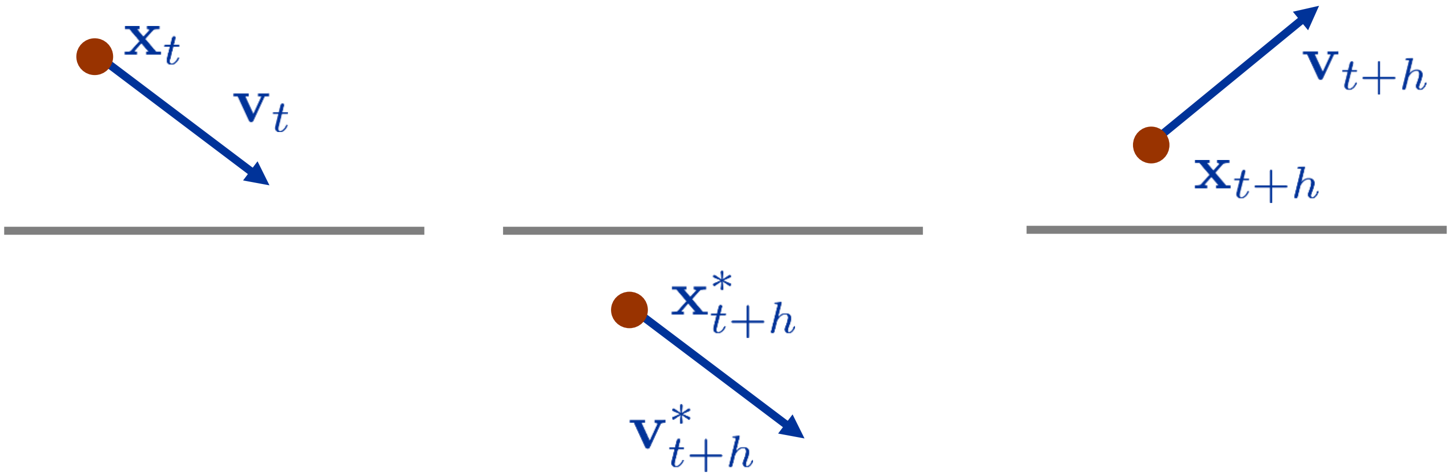
UNI
FREIBURG

Collision Handling at Planes

after time integration

after collision handling

e.g., reflect predicted position and velocity at the collision plane



current state at time t

predicted state at time t+h

final state at time t+h

Euler with Collision Handling

- predict position and velocity at t+h

$$\mathbf{x}^*_{t+h} = \mathbf{x}_t + h\mathbf{v}_t$$
$$\mathbf{v}^*_{t+h} = \mathbf{v}_t + h\mathbf{a}_t$$

- collision detection
- collision handling

$$\mathbf{x}_{t+h} = \text{reflect } \mathbf{x}^*_{t+h}$$
$$\mathbf{v}_{t+h} = \text{reflect } \mathbf{v}^*_{t+h}$$

Heun with Collision Handling

- predict position and velocity at t+h

  $\mathbf{x}^*_{t+h} = \mathbf{x}_t + h\mathbf{v}_t$

  $\mathbf{v}^*_{t+h} = \mathbf{v}_t + h\mathbf{a}_t$
- collision detection
- collision handling

  $\mathbf{x}^{**}_{t+h} = \text{reflect } \mathbf{x}^*_{t+h}$

  $\mathbf{v}^{**}_{t+h} = \text{reflect } \mathbf{v}^*_{t+h}$
- predict acceleration at t+h

  $\mathbf{a}^{**}_{t+h}(\mathbf{x}^{**}_{t+h}, \mathbf{v}^{**}_{t+h})$
- compute final position and velocity

  $\mathbf{x}_{t+h} = \mathbf{x}_t + \frac{h}{2}(\mathbf{v}_t + \mathbf{v}^{**}_{t+h})$ $\mathbf{v}_{t+h} = \mathbf{v}_t + \frac{h}{2}(\mathbf{a}_t + \mathbf{a}^{**}_{t+h})$

Verlet with Collision Handling

- $\mathbf{x}_{t+h} = 2\mathbf{x}_t - \mathbf{x}_{t-h} + h^2\mathbf{a}_t$
- How to reflect the velocity in case of a collision?
- Solution: rewrite Verlet using a velocity approximation
- $\mathbf{v}_t = \frac{1}{h}(\mathbf{x}_t - \mathbf{x}_{t-h})$
- $\mathbf{x}_{t+h} = \mathbf{x}_t + \frac{h}{h}(\mathbf{x}_t - \mathbf{x}_{t-h}) + h^2\mathbf{a}_t = \mathbf{x}_t + h\mathbf{v}_t + h^2\mathbf{a}_t$
- Now, the standard concept can be used

  $\mathbf{x}^*_{t+h} = \mathbf{x}_t + h\mathbf{v}_t + h^2\mathbf{a}_t$

  $\mathbf{v}^*_{t+h} = \frac{1}{h}(\mathbf{x}^*_{t+h} - \mathbf{x}_t)$
- collision handling

  $\mathbf{x}_{t+h} = \text{reflect } \mathbf{x}^*_{t+h}$

  $\mathbf{v}_{t+h} = \text{reflect } \mathbf{v}^*_{t+h}$

Implicit Euler – Mass-Spring-System

- for a particle i

$$\mathbf{x}_i^{t+h} = \mathbf{x}_i^t + h\mathbf{v}_i^{t+h}$$
$$\mathbf{v}_i^{t+h} = \mathbf{v}_i^t + h\frac{1}{m}\mathbf{F}_i^{t+h}(\mathbf{x}_i^{t+h}, \ldots, \mathbf{x}_j^{t+h})$$

- for a set of particles

$$\mathbf{x}^t = (\mathbf{x}_1^{\mathrm{T}}, \mathbf{x}_2^{\mathrm{T}}, \ldots, \mathbf{x}_n^{\mathrm{T}})^{\mathrm{T}}$$
$$\mathbf{v}^t = (\mathbf{v}_1^{\mathrm{T}}, \mathbf{v}_2^{\mathrm{T}}, \ldots, \mathbf{v}_n^{\mathrm{T}})^{\mathrm{T}}$$
$$\mathbf{F}^{\mathbf{t}}(\mathbf{x}^t) = (\mathbf{F}_1^{\mathrm{T}}, \mathbf{F}_2^{\mathrm{T}}, \ldots, \mathbf{F}_n^{\mathrm{T}})^{\mathrm{T}}$$
$$\mathbf{M} = diag(m_1, m_1, m_1, \ldots, m_n, m_n, m_n) \in \mathbb{R}^{3n \times 3n}$$

$$\mathbf{x}^{t+h} = \mathbf{x}^t + h\mathbf{v}^{t+h}$$
$$\mathbf{v}^{t+h} = \mathbf{v}^t + h\mathbf{M}^{-1}\mathbf{F}^{t+h}(\mathbf{x}^{t+h})$$

Example

- three particles

$$\mathbf{x}^t = (x^t_{1,x}, x^t_{1,y}, x^t_{1,z}, x^t_{2,x}, x^t_{2,y}, x^t_{2,z}, x^t_{3,x}, x^t_{3,y}, x^t_{3,z})^{\mathrm{T}}$$

$$\mathbf{v}^t = (v^t_{1,x}, v^t_{1,y}, v^t_{1,z}, v^t_{2,x}, v^t_{2,y}, v^t_{2,z}, v^t_{3,x}, v^t_{3,y}, v^t_{3,z})^{\mathrm{T}}$$

$$\mathbf{F}^{\mathbf{t}}(\mathbf{x}^t) = (F^t_{1,x}, F^t_{1,y}, F^t_{1,z}, F^t_{2,x}, F^t_{2,y}, F^t_{2,z}, F^t_{3,x}, F^t_{3,y}, F^t_{3,z})^{\mathrm{T}}$$

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_3 \end{pmatrix}$$

$$\mathbf{x}^{t+h} = \mathbf{x}^t + h\mathbf{v}^{t+h}$$

$$\mathbf{v}^{t+h} = \mathbf{v}^t + h\mathbf{M}^{-1}\mathbf{F}^{t+h}(\mathbf{x}^{t+h})$$

Force Linearization

- We have to solve a system to compute v at t+h

$$\mathbf{v}^{t+h} = \mathbf{v}^t + h\mathbf{M}^{-1}\mathbf{F}^{t+h}(\mathbf{x}^{t+h})$$

$$\mathbf{M}\mathbf{v}^{t+h} = \mathbf{M}\mathbf{v}^t + h\mathbf{F}^{t+h}(\mathbf{x}^{t+h})$$

$$\mathbf{M}\mathbf{v}^{t+h} = \mathbf{M}\mathbf{v}^t + h\mathbf{F}^{t+h}(\mathbf{x}^t + h\mathbf{v}^{t+h})$$

- force linearization

$$\mathbf{J}^t(\mathbf{x}^t) = \frac{\partial \mathbf{F}^t}{\partial \mathbf{x}^t} \in \mathbb{R}^{3n\times 3n}$$

$$\mathbf{M}\mathbf{v}^{t+h} = \mathbf{M}\mathbf{v}^t + h(\mathbf{F}^t(\mathbf{x}^t) + \mathbf{J}^t(\mathbf{x}^t)h\mathbf{v}^{t+h})$$

$$\mathbf{M}\mathbf{v}^{t+h} = \mathbf{M}\mathbf{v}^t + h\mathbf{F}^t(\mathbf{x}^t) + h^2\mathbf{J}^t(\mathbf{x}^t)\mathbf{v}^{t+h}$$

$$\left(\mathbf{M} - h^2\mathbf{J}^t(\mathbf{x}^t)\right)\mathbf{v}^{t+h} = \mathbf{M}\mathbf{v}^t + h\mathbf{F}^t(\mathbf{x}^t)$$

Jacobian

- In the Jacobian $\mathbf{J}^t$, a spring force between $\mathbf{x}_i^t$ and $\mathbf{x}_j^t$ is represented by four sub matrices
  $\mathbf{J}^t_{i,j} \in \mathbb{R}^{3\times 3},\ \mathbf{J}^t_{j,i} \in \mathbb{R}^{3\times 3},\ \mathbf{J}^t_{i,i} \in \mathbb{R}^{3\times 3},\ \mathbf{J}^t_{j,j} \in \mathbb{R}^{3\times 3}$
  that are accumulated at positions
  $(3i,3j),\ (3j,3i),\ (3i,3i),\ (3j,3j)$

$$\mathbf{J}^t_{i,i} = \frac{\partial \mathbf{F}_i^t}{\partial \mathbf{x}_i^t} \in \mathbb{R}^{3\times 3}$$

$$\begin{aligned}\mathbf{J}^t_{i,i} &= \frac{\partial}{\partial \mathbf{x}_i^t} k_s \left((\mathbf{x}_j - \mathbf{x}_i) - L_s \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \right) \\ &= k_s \left(-\mathbf{I} + \frac{L_s}{|\mathbf{x}_j - \mathbf{x}_i|} \left(\mathbf{I} - \frac{1}{|\mathbf{x}_j - \mathbf{x}_i|^2} (\mathbf{x}_j - \mathbf{x}_i)(\mathbf{x}_j - \mathbf{x}_i)^{\mathrm{T}} \right) \right) \\ &= -\mathbf{J}^t_{i,j} = \mathbf{J}^t_{j,j} = -\mathbf{J}^t_{j,i}\end{aligned}$$

Example

- three particles $\mathbf{x}_1^t\ \mathbf{x}_2^t\ \mathbf{x}_3^t$
- two springs connecting $\mathbf{x}_1^t\ \mathbf{x}_2^t$ and $\mathbf{x}_2^t\ \mathbf{x}_3^t$

$$\mathbf{J}_{\mathbf{x}^t}^t = \begin{pmatrix} \frac{\partial F_{x_{1,x}}^t}{\partial x_{1,x}^t} & \cdots & \frac{\partial F_{x_{1,x}}^t}{\partial x_{3,z}^t} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_{x_{3,z}}^t}{\partial x_{1,x}^t} & \cdots & \frac{\partial F_{x_{3,z}}^t}{\partial x_{3,z}^t} \end{pmatrix}$$

$$\mathbf{J}_{\mathbf{x}^t}^t = \begin{pmatrix} \mathbf{J}_{1,1}^t & \mathbf{J}_{1,2}^t & \mathbf{J}_{1,3}^t \\ \mathbf{J}_{2,1}^t & \mathbf{J}_{2,2}^t & \mathbf{J}_{2,3}^t \\ \mathbf{J}_{3,1}^t & \mathbf{J}_{3,2}^t & \mathbf{J}_{3,3}^t \end{pmatrix} \qquad \mathbf{J}_{i,i}^t = \frac{\partial \mathbf{F}_i^t}{\partial \mathbf{x}_i^t} \in \mathbb{R}^{3\times 3}$$

Example

- force $\mathbf{F}_3^t$ depends on spring 2, i.e. positions $\mathbf{x}_2^t\ \mathbf{x}_3^t$
- force $\mathbf{F}_3^t$ does not depend on position $\mathbf{x}_1^t$
- similarly, $\mathbf{F}_1^t$ does not depend on position $\mathbf{x}_3^t$
- therefore,

$$\mathbf{J}_{3,1}^t = \mathbf{J}_{1,3}^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- and

$$\mathbf{J}_{\mathbf{x}^t}^t = \begin{pmatrix} \mathbf{J}_{1,1}^t & \mathbf{J}_{1,2}^t & \mathbf{0} \\ \mathbf{J}_{2,1}^t & \mathbf{J}_{2,2}^t & \mathbf{J}_{2,3}^t \\ \mathbf{0} & \mathbf{J}_{3,2}^t & \mathbf{J}_{3,3}^t \end{pmatrix}$$

Example

- $\mathbf{J}^t_{i,j}$ has to be computed, if $\mathbf{F}^t_i$ depends on $\mathbf{x}^t_j$
- $\mathbf{J}^t_{i,j}$ has nine components
- e.g., $J^t_{1,x,1,x} = \frac{\partial F_{1,x}}{\partial x_{1,x}}$

$$F^t_{1,x} = k_1(x_{2,x} - x_{1,x} - L_1 \frac{x_{2,x}-x_{1,x}}{\sqrt{(x_{2,x}-x_{1,x})^2+(x_{2,y}-x_{1,y})^2+(x_{2,z}-x_{1,z})^2}})$$

$$\frac{\partial F_{1,x}}{\partial x_{1,x}} = k_1 \left(-1 - L_1 \frac{-1\sqrt{\dots}-(x_{2,x}-x_{1,x})\frac{1}{2\sqrt{\dots}}2(x_{2,x}-x_{1,x})(-1)}{\sqrt{\dots}^2}\right)$$

$$\frac{\partial F_{1,x}}{\partial x_{1,x}} = k_1 \left(-1 + L_1(\frac{1}{\sqrt{\dots}} - \frac{(x_{2,x}-x_{1,x})^2}{\sqrt{\dots}^3})\right)$$

$$\frac{\partial F_{1,x}}{\partial x_{1,x}} = k_1 \left(-1 + \frac{L_1}{\sqrt{\dots}}(1 - \frac{(x_{2,x}-x_{1,x})^2}{\sqrt{\dots}^2})\right)$$

Example

- $\frac{\partial F_{1,x}}{\partial x_{1,x}} = k_1 \left(-1 + \frac{L_1}{\sqrt{\dots}} (1 - \frac{(x_{2,x} - x_{1,x})^2}{\sqrt{\dots}^2}) \right)$

  $\mathbf{J}_{i,i}^t = \frac{\partial}{\partial \mathbf{x}_i^t} k_s \left((\mathbf{x}_j - \mathbf{x}_i) - L_s \frac{\mathbf{x}_j - \mathbf{x}_i}{|\mathbf{x}_j - \mathbf{x}_i|} \right)$

  $= k_s \left(-\mathbf{I} + \frac{L_s}{|\mathbf{x}_j - \mathbf{x}_i|} \left(\mathbf{I} - \frac{1}{|\mathbf{x}_j - \mathbf{x}_i|^2} (\mathbf{x}_j - \mathbf{x}_i)(\mathbf{x}_j - \mathbf{x}_i)^{\mathrm{T}} \right) \right)$

  $= -\mathbf{J}_{i,j}^t = \mathbf{J}_{j,j}^t = -\mathbf{J}_{j,i}^t$