Simulation in Computer Graphics

Exercises - Notes

Matthias Teschner

Computer Science Department
University of Freiburg

Elastic Solids

- displacement
- strain
- stress
- force

Elastic Solids

- deformation map
- deformation gradient
- strain tensor
- stress tensor
- force

Deformation Map and Deformation Gradient

- deformation map
  - relation between original (undeformed) $\mathbf{X}$ and current (deformed) location $\mathbf{x}$
    $\mathbf{x} = \phi(\mathbf{X})$
- deformation gradient tensor
  - Jacobian of ϕ

$$\mathbf{F} = \frac{\partial \phi}{\partial \mathbf{X}} = \begin{pmatrix} \frac{\partial \phi_1}{\partial X_1} & \frac{\partial \phi_1}{\partial X_2} & \frac{\partial \phi_1}{\partial X_3} \\ \frac{\partial \phi_2}{\partial X_1} & \frac{\partial \phi_2}{\partial X_2} & \frac{\partial \phi_2}{\partial X_3} \\ \frac{\partial \phi_3}{\partial X_1} & \frac{\partial \phi_3}{\partial X_2} & \frac{\partial \phi_3}{\partial X_3} \end{pmatrix}$$

Deformation Map and Deformation Gradient

- examples
- $\phi(\mathbf{X}) = \mathbf{X} + \mathbf{t} \qquad \mathbf{F} = \mathbf{I}$
- $\phi(\mathbf{X}) = \gamma\mathbf{X} \qquad \mathbf{F} = \gamma\mathbf{I}$
- $$\phi(\mathbf{X}) = \begin{pmatrix} 2X_1 \\ 3X_2 \\ 4X_3 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$
- $$\phi(\mathbf{X}) = \begin{pmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} \cos\pi & -\sin\pi & 0 \\ \sin\pi & \cos\pi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

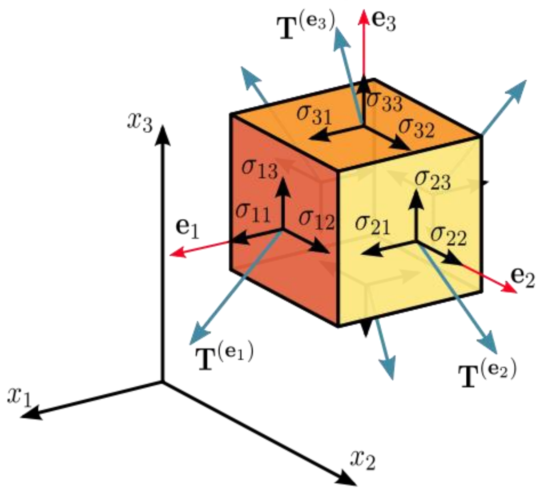
Strain

- Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$
- translated object $\mathbf{F} = \mathbf{I} \quad \mathbf{E} = 0$
- rotated object $\mathbf{F} = \mathbf{R} \quad \mathbf{E} = 0$
- deformed object $\mathbf{F} = \mathbf{RS} \quad \mathbf{E} = \frac{1}{2}(\mathbf{S}^2 - \mathbf{I})$

- small (infinitesimal) strain tensor $\epsilon = \frac{1}{2}(\mathbf{F}^T + \mathbf{F}) - \mathbf{I}$
- faster to compute
- works only for small deformations without rotations

Stress

- stress arises from / tends to change
  - volume deviations
  - distortions
- volumetric / mean normal stress
  - $\sigma_{11}, \sigma_{22}, \sigma_{33}$ contribute to forces orthogonal to the surface
  - orthogonal forces lead to volume changes
- deviatoric stress
  - forces parallel to the surface correspond to distortions



Wikipedia

Splitting Stress into Volumetric and Deviatoric

- overall stress

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

- volumetric stress

$$\sigma_v = \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix} \quad \pi = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}$$

- deviatoric stress

$$\sigma_d = \sigma - \sigma_v = \begin{pmatrix} \sigma_{11} - \pi & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \pi & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \pi \end{pmatrix}$$

The diagonal is not necessarily equal to zero. The trace, however, i.e. the sum of the diagonal elements is equal to zero.

Volumetric vs. Deviatoric Stress

- examples

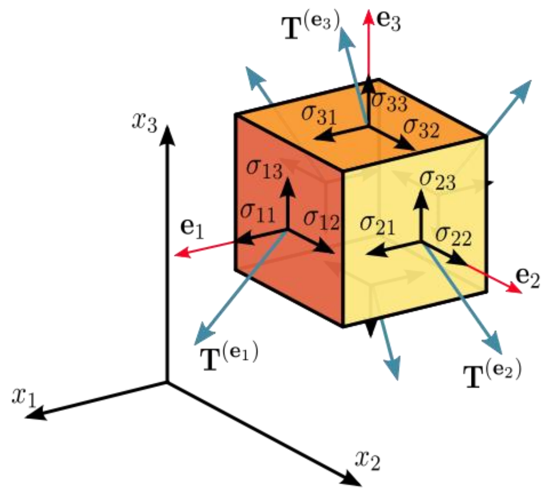
$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \sigma_v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \sigma_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \sigma_v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \sigma_d = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- diagonal elements contribute to volumetric or deviatoric stress
- non-diagonal elements exclusively contribute to deviatoric stress

Surface Force

- the stress tensor describes the force distribution on the surface of a volume (with constant stress)
- traction $\mathbf{T}$ on a surface with normal $\mathbf{n}$
  $\mathbf{T}(\mathbf{n}) = \sigma \cdot \mathbf{n}$
- e.g.: $\mathbf{T}(\mathbf{e_1}) = \sigma \cdot \mathbf{e_1} = (\sigma_{11}, \sigma_{21}, \sigma_{31})^{\mathrm{T}}$
- surface force $\mathbf{F}$ at a surface with normal $\mathbf{n}$ and area A
  $\mathbf{F}(\mathbf{n}) = A \cdot \sigma \cdot \mathbf{n}$



Wikipedia

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Constitutive Model (Strain-Stress Relation)

- compute stress from strain

$$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} \longrightarrow \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

- e.g., Hooke's law for isotropic material

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix}$$

- Young's modulus E (measure of stretch)
- Poisson ratio ν (measure of incompressibility)

Deformation Map and Deformation Gradient of a Tetrahedron

- undeformed state $\mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$
- current state $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$
- translation does not influence strain
  - translate both states so that $\mathbf{x}_0 = \mathbf{X}_0 = 0$
- $\mathbf{X} = \mathbf{X}_1 b_1 + \mathbf{X}_2 b_2 + \mathbf{X}_3 b_3 = [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]\mathbf{b}$
- $\mathbf{b} = [\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]^{-1}\mathbf{X}$
- $\mathbf{x} = \mathbf{x}_1 b_1 + \mathbf{x}_2 b_2 + \mathbf{x}_3 b_3 = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]\mathbf{b}$
- $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]^{-1}\mathbf{X}$
- $\phi(\mathbf{X}) = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]^{-1}\mathbf{X}$
- $\mathbf{F} = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3][\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3]^{-1}$

Strain, Stress per Tetrahedron
Force per Face and per Vertex

- strain $\epsilon = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$
- stress $\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{pmatrix}$
- force at a face with normal $\mathbf{n}$ and area A

  $\mathbf{F}_{i,j,k} = A_{i,j,k} \cdot \sigma \cdot \mathbf{n}_{i,j,k} = \frac{1}{2}\sigma[(\mathbf{x}_j - \mathbf{x}_i) \times (\mathbf{x}_k - \mathbf{x}_i)]$
- forces at vertices

  $\mathbf{F}_i = \mathbf{F}_i + \frac{1}{3}\mathbf{F}_{i,j,k}$

  $\mathbf{F}_j = \mathbf{F}_j + \frac{1}{3}\mathbf{F}_{i,j,k}$

  $\mathbf{F}_k = \mathbf{F}_k + \frac{1}{3}\mathbf{F}_{i,j,k}$

Literature

- Sifakis, Barbic: FEM Simulation of 3D Deformable Solids: A practitioner's guide to theory, discretization and model reduction, SIGGRAPH 2012 course, http://femdefo.org
- David Raymond, Introduction to Continuum Mechanics, 1999. http://kestrel.nmt.edu/~raymond/classes/ph536/continuum.pdf