MLS Pressure Boundaries for Divergence-Free and Viscous SPH Fluids - Appendix

Stefan Band

1 Mass of a boundary particle

Iterative solvers such as PCISPH [11], IISPH [5] or DFSPH [4] compute a pressure field p and apply pressure accelerations of the form

$$\boldsymbol{a}_f^{\mathrm{p}} = -\sum_j m_j \left(\frac{p_f}{\rho_f^2} + \frac{p_j}{\rho_j^2} \right) \boldsymbol{\nabla} W_{fj} - \sum_b m_b \left(\frac{p_f}{\rho_f^2} + \frac{p_b}{\rho_b^2} \right) \boldsymbol{\nabla} W_{ff_b} \tag{1}$$

to fluid particles f [3]. Here, j and b denote fluid and boundary neighbors of fluid particle f, respectively. Equation (1) requires a notion of mass m_b and density ρ_b at boundary particles b.

The mass m_b at a boundary particle b can be geometrically motivated from the boundary particle volume V_b [1, 10, 2], i.e. the mass m_b can be derived from the relation $m_b = \rho_b V_b$. Since the density ρ_b is typically set to the rest density of the adjacent fluid particle [1], i.e.

$$\rho_b = \rho_f \ , \tag{2}$$

the mass m_b of a boundary particle b can be computed as

$$m_b = \rho_f^0 V_b^0 \ , \tag{3}$$

where ρ_f^0 denotes the rest density of fluid particle f and the rest volume V_b^0 is [2]

$$V_b^0 = \frac{\gamma}{\sum_{b_b} W_{bb_b}} \ . \tag{4}$$

The computation of Eq. (4) only processes boundary neighbors b_b of a boundary particle b as the rest volume does not depend on possibly adjacent fluid particles [2]. As only one layer of boundary particles is used to represent the surface of the boundary [1], the coefficient γ accounts for an incomplete neighborhood. The coefficient depends on the choice of the SPH kernel function. For the cubic spline kernel [7] with a smoothing length of two times the particle size h, $\gamma \approx 0.7$ [2]. This motivated by the fact that $0.7/\sum_{b_b} W_{bb_b} = h^3$ for boundary particles that are evenly sampled in a plane. See Fig. 1 for an illustration of the derivation.

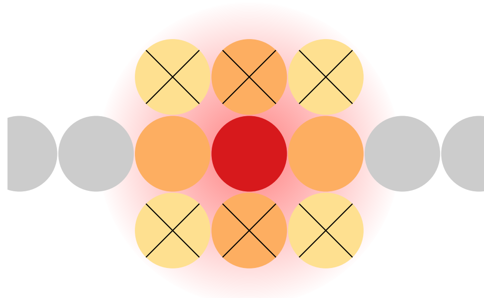

Figure 1: Cross section of a boundary plane with color-coded SPH kernel weights. In 3D, red represents 31.8 %, orange 8 % and yellow 1.6 % of the total kernel weight. Due to missing particle neighbors (crossed), the sum of kernel weights of the central particle is not 100 % but $1 \cdot 31.8\,\% + 4 \cdot 8\,\% + 4 \cdot 1.6\,\% = 70.2\,\%$.

Implementation Employing Eqs. (2) to (4), the pressure acceleration of a fluid particle f, Eq. (1), can be computed as

$$\boldsymbol{a}_f^{\mathrm{p}} = -\sum_j m_j \left(\frac{p_f}{\rho_f^2} + \frac{p_j}{\rho_j^2} \right) \boldsymbol{\nabla} W_{fj} - \sum_b \rho_f^0 V_b^0 \left(\frac{p_f}{\rho_f^2} + \frac{p_b}{\rho_f^2} \right) \boldsymbol{\nabla} W_{fb} \,. \tag{5}$$

2 Pressure Extrapolation with MLS

The pressure p_b at boundary particle b is computed as

$$p_b = \left(1, \bar{x}_b, \bar{y}_b, \bar{z}_b\right)^{\mathrm{T}} \cdot \boldsymbol{c}_b \,, \tag{6}$$

where $\boldsymbol{c}_b = \left(\alpha_b, \beta_b, \gamma_b, \delta_b\right)^{\mathrm{T}}$ is a vector of unknown coefficients. The first parameter α_b is the weighted average of the pressure values of fluid particles b_f adjacent to boundary particle b:

$$\alpha_b = \frac{\sum_{b_f} p_{b_f} V_{b_f} W_{bb_f}}{\sum_{b_f} V_{b_f} W_{bb_f}} \,, \tag{7}$$

The other parameters β_b, γ_b and δ_b are obtained by solving

$$\begin{bmatrix} \beta_b \\ \gamma_b \\ \delta_b \end{bmatrix} = \left(\sum_{b_f} \begin{bmatrix} \bar{x}_{b_f}^2 & \bar{x}_{b_f}\bar{y}_{b_f} & \bar{x}_{b_f}\bar{z}_{b_f} \\ \bar{x}_{b_f}\bar{y}_{b_f} & \bar{y}_{b_f}^2 & \bar{y}_{b_f}\bar{z}_{b_f} \\ \bar{x}_{b_f}\bar{z}_{b_f} & \bar{y}_{b_f}\bar{z}_{b_f} & \bar{z}_{b_f}^2 \end{bmatrix} V_{b_f} W_{bb_f} \right)^{-1} \sum_{b_f} \begin{bmatrix} \bar{x}_{b_f} \\ \bar{y}_{b_f} \\ \bar{z}_{b_f} \end{bmatrix} p_{b_f} V_{b_f} W_{bb_f} \,. \tag{8}$$

Implementation In our experiments, we experienced issues with a singular matrix in Eq. (8) for boundary particles whose fluid neighbors are co-linear or co-planar. In these cases, we follow [8] and use safe inversion via Singular Value Decomposition (SVD) [9] to avoid the problems with singular matrices. We use the SVD implementation of [6].

Also, we set β_b, γ_b and δ_b to zero if $|| \sum_{b_f} \left(\bar{x}_{b_f}, \bar{y}_{b_f}, \bar{z}_{b_f}\right)^{\mathrm{T}} p_{b_f} V_{b_f} W_{bb_f} || < \epsilon$, where ϵ is a small value, e.g., 10^{-5}. This results in a pressure value $p_b = \alpha_b$.

References

[1] N. Akinci, M. Ihmsen, G. Akinci, B. Solenthaler, and M. Teschner. Versatile Rigid-fluid Coupling for Incompressible SPH. *ACM Transactions on Graphics*, 31(4):62:1–62:8, 2012.

[2] S. Band, C. Gissler, M. Ihmsen, J. Cornelis, A. Peer, and M. Teschner. Pressure Boundaries for Implicit Incompressible SPH. *ACM Transactions on Graphics*, 37(2):14:1–14:11, 2018. Presented at SIGGRAPH 2018.

[3] S. Band, C. Gissler, A. Peer, and M. Teschner. MLS Pressure Boundaries for Divergence-Free and Viscous SPH Fluids. *Computers & Graphics*, 76:37–46, 2018.

[4] J. Bender and D. Koschier. Divergence-Free SPH for Incompressible and Viscous Fluids. *IEEE Transactions on Visualization and Computer Graphics*, 23(3):1193–1206, 2017.

[5] M. Ihmsen, J. Cornelis, B. Solenthaler, C. Horvath, and M. Teschner. Implicit incompressible SPH. *IEEE Transactions on Visualization and Computer Graphics*, 20(3):426–435, 2014.

[6] Jacob, Benoît and Guennebaud, Gaël. Eigen. `https://eigen.tuxfamily.org`, 2018.

[7] J. J. Monaghan. Smoothed Particle Hydrodynamics. *Reports on Progress in Physics*, 68(8):1703, 2005.

[8] M. Müller, R. Keiser, A. Nealen, M. Pauly, M. H. Gross, and M. Alexa. Point Based Animation of Elastic, Plastic and Melting Objects. In *ACM SIGGRAPH/Eurographics Symposium on Computer Animation*, pages 141–151. Eurographics Association, 2004.

[9] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes 3rd Edition: The Art of Scientific Computing.* Cambridge University Press, New York, NY, USA, 3 edition, 2007.

[10] S. Rosswog. SPH Methods in the Modelling of Compact Objects. *Living Reviews in Computational Astrophysics*, 1(1), 2015.

[11] B. Solenthaler and R. Pajarola. Predictive-corrective Incompressible SPH. *ACM Transactions on Graphics*, 28(3):40:1–40:6, 2009.