Interlinked SPH Pressure Solvers for Strong Fluid-Rigid Coupling

Appendix

Christoph Gissler

Contents

B Diagonal element computation

The goal of this document is to explain the computation of the diagonal element b_r in more detail. All references that do not start with "B" are references to formulas or sections in the main manuscript. Intuitively, b_r is the coefficient which relates a pressure increase of particle r to a predicted density change of the same particle.

Mathematically, the diagonal element b_r can be derived by extracting all the coefficients of p_r from Eq. (8):

$$s_r = \rho_r \boldsymbol{\nabla} \cdot \underbrace{\left(\Delta t \sum_{r_r} V_{r_r} \mathbf{K}_{r r_r} \boldsymbol{\nabla} p_{r_r} \right)}_{-\mathbf{v}_r^{\mathrm{rr}}}, \tag{8 revisited}$$

where r_r are particles of the same rigid R as r.

As shown in Subsection 3.4.2, we can compute the right-hand side of Eq. (8) using SPH as:

$$-\sum_{r_k} V_{r_k} \rho_{r_k} (\mathbf{v}_{r_k}^{\mathrm{rr}} - \mathbf{v}_r^{\mathrm{rr}}) \cdot \boldsymbol{\nabla} W_{r r_k},$$

where r_k are neighboring rigid particles of r of other rigid bodies K.

By using the formula for $\mathbf{v}^{\mathrm{rr}}_{r_k}$ and $\mathbf{v}^{\mathrm{rr}}_{r}$ based on Eq. (8), we get

$$-\sum_{r_k} V_{r_k}\rho_{r_k}\left[\left(\Delta t\sum_{r_{k_r}} V_{r_{k_r}}\mathbf{K}_{r_k r_{k_r}}\,\boldsymbol{\nabla}\, p_{r_{k_r}}\right)-\left(\Delta t\sum_{r_r} V_{r_r}\mathbf{K}_{r r_r}\,\boldsymbol{\nabla}\, p_{r_r}\right)\right]\cdot\boldsymbol{\nabla} W_{r r_k}, \tag{B.1}$$

where r_k are neighboring particles of r belonging to other rigid bodies K, r_{k_r} are rigid particles of the same rigid body K as r_k and r_r are rigid particles of the same rigid body R as r.

As described in Subsection 3.4.2, we calculate the pressure gradient as:

$$\boldsymbol{\nabla}\, p_r = \rho_r\sum_{r_k} V_{r_k}\rho_{r_k}\left(\frac{p_r}{\rho_r^2}+\frac{p_{r_k}}{\rho_{r_k}^2}\right)\boldsymbol{\nabla} W_{r r_k}, \tag{B.2}$$

where r_k are neighboring rigid particles of r belonging to other rigid bodies K.

Using Eq. (B.2) in Eq. (B.1), we get:

$$\begin{aligned}-\sum_{r_k} V_{r_k}\rho_{r_k}\Bigg[\Bigg(\Delta t\sum_{r_{k_r}} V_{r_{k_r}}\mathbf{K}_{r_k r_{k_r}}\Bigg[\rho_{r_{k_r}}\sum_{r_{k_{r_k}}} V_{r_{k_{r_k}}}\rho_{r_{k_{r_k}}}\left(\frac{p_{r_{k_r}}}{\rho_{r_{k_r}}^2}+\frac{p_{r_{k_{r_k}}}}{\rho_{r_{k_{r_k}}}^2}\right)\boldsymbol{\nabla} W_{r_{k_r} r_{k_{r_k}}}\Bigg]\Bigg)\\ -\Bigg(\Delta t\sum_{r_r} V_{r_r}\mathbf{K}_{r r_r}\Bigg[\rho_{r_r}\sum_{r_{r_k}} V_{r_{r_k}}\rho_{r_{r_k}}\left(\frac{p_{r_r}}{\rho_{r_r}^2}+\frac{p_{r_{r_k}}}{\rho_{r_{r_k}}^2}\right)\boldsymbol{\nabla} W_{r_r r_{r_k}}\Bigg]\Bigg)\Bigg]\cdot\boldsymbol{\nabla} W_{r r_k}.\end{aligned} \tag{B.3}$$

In Eq. (B.3), $r_{k_{r_k}}$ are neighboring particles of r_{k_r} which belong to other rigid bodies than r_{k_r}.

To get the coefficients of p_r, we can remove all the pressure terms where p_r cannot occur. These are the fractions including $p_{r_{k_r}}$ (since these only include all particles of a rigid body K of neighboring particles r_k of particle r) and $p_{r_{r_k}}$ (since these are neighboring particles belonging to other rigid bodies K).

$$\begin{aligned}-\sum_{r_k} V_{r_k}\rho_{r_k}\Bigg[\Bigg(\Delta t\sum_{r_{k_r}} V_{r_{k_r}}\mathbf{K}_{r_k r_{k_r}}\Bigg[\rho_{r_{k_r}}\sum_{r_{k_{r_k}}} V_{r_{k_{r_k}}}\rho_{r_{k_{r_k}}}\left(\frac{p_{r_{k_{r_k}}}}{\rho_{r_{k_{r_k}}}^2}\right)\boldsymbol{\nabla} W_{r_{k_r} r_{k_{r_k}}}\Bigg]\Bigg)\\ -\Bigg(\Delta t\sum_{r_r} V_{r_r}\mathbf{K}_{r r_r}\Bigg[\rho_{r_r}\sum_{r_{r_k}} V_{r_{r_k}}\rho_{r_{r_k}}\left(\frac{p_{r_r}}{\rho_{r_r}^2}\right)\boldsymbol{\nabla} W_{r_r r_{r_k}}\Bigg]\Bigg)\Bigg]\cdot\boldsymbol{\nabla} W_{r r_k}\end{aligned} \tag{B.4}$$

We replace the remaining pressure terms with p_r and also adapt the associated values:

$$
\begin{aligned}
-\sum_{r_k} V_{r_k}\rho_{r_k} \Bigg[\Bigg(\Delta t \sum_{r_{k_r}} V_{r_{k_r}} \mathbf{K}_{r_k r_{k_r}} \Bigg[\rho_{r_{k_r}} \sum_{r \in r_{k_{r_k}}} V_r \rho_r \left(\frac{p_r}{\rho_r^2} \right) \boldsymbol{\nabla} W_{r_{k_r} r} \Bigg] \Bigg) \\
- \Bigg(\Delta t V_r \mathbf{K}_{rr} \Bigg[\rho_r \sum_{r_k} V_{r_k} \rho_{r_k} \left(\frac{p_r}{\rho_r^2} \right) \boldsymbol{\nabla} W_{r r_k} \Bigg] \Bigg) \Bigg] \cdot \boldsymbol{\nabla} W_{r r_k}.
\end{aligned}
\tag{B.5}
$$

Finally, we move p_r out of the term to get the diagonal b_r:

$$
\begin{aligned}
b_r = -\sum_{r_k} V_{r_k}\rho_{r_k} \Bigg[\Bigg(\Delta t \sum_{r_{k_r}} V_{r_{k_r}} \mathbf{K}_{r_k r_{k_r}} \Bigg[\rho_{r_{k_r}} \sum_{r \in r_{k_{r_k}}} V_r \left(\frac{1}{\rho_r} \right) \boldsymbol{\nabla} W_{r_{k_r} r} \Bigg] \Bigg) \\
- \Bigg(\Delta t V_r \mathbf{K}_{rr} \Bigg[\sum_{r_k} V_{r_k} \rho_{r_k} \left(\frac{1}{\rho_r} \right) \boldsymbol{\nabla} W_{r r_k} \Bigg] \Bigg) \Bigg] \cdot \boldsymbol{\nabla} W_{r r_k}.
\end{aligned}
\tag{B.6}
$$

On a first glance, it seems computationally expensive to compute b_r using Eq. (B.6) for each rigid particle. However, similar to how the right-hand side of Eq. (8) can be computed by two loops over all rigid particles as described in Subsection 3.4.2, we can compute the diagonal for all rigid particles in a single loop over all particles. Algorithm B.1 shows an overview of the necessary steps.

1: **foreach** particle r of rigid body R **do**
2: Compute gradient $\boldsymbol{\nabla} p_r^b = \rho_r \sum_{r_k} V_{r_k}\rho_{r_k} \frac{1}{\rho_r^2} \boldsymbol{\nabla} W_{r r_k}$
3: Compute linear $\mathbf{v}_R^b$ and angular velocity $\boldsymbol{\omega}_R^b$ of R using $\boldsymbol{\nabla} p_r^b$
4: Compute particle velocity $\mathbf{v}_r^b$ using $\mathbf{v}_R^b$ and $\boldsymbol{\omega}_R^b$
5: **foreach** neighboring rigid K **do**
6: Compute pairwise $\boldsymbol{\nabla} p_{r_k r}^b = \rho_{r_k} V_r \rho_r \frac{1}{\rho_r^2} \boldsymbol{\nabla} W_{r_k r}$ for all neighbors r_k of r
7: Compute $\mathbf{v}_K^b$ and $\boldsymbol{\omega}_K^b$ of K using $\boldsymbol{\nabla} p_{r_k r}^b$ of all neighbors r_k of r
8: Compute divergence $\sum_{r_k \in K} V_{r_k}\rho_{r_k}(\mathbf{v}_{r_k}^b - \mathbf{v}_r^b) \cdot \boldsymbol{\nabla} W_{r r_k}$ where $\mathbf{v}_{r_k}^b$ is computed on-the-fly using $\mathbf{v}_K^b$ and $\boldsymbol{\omega}_K^b$
9: $b_r = -\sum_{r_k \in K} V_{r_k}\rho_{r_k}(\mathbf{v}_{r_k}^b - \mathbf{v}_r^b) \cdot \boldsymbol{\nabla} W_{r r_k}$ for all K using sub-sums from Line 8

Algorithm B.1: Computing the diagonal element of particle r of rigid body R with neighboring rigid bodies K.

In praxis, we observed that for the calculation of b_r it is in most scenarios not necessarily important to consider the changed velocities of the neighboring rigid bodies based on the pressure increase of r. Accordingly, Lines 6 and 7 in Algorithm B.1 could be skipped. Instead, in Line 8, the velocity of the neighboring rigid K is assumed to be unchanged by the pressure of r. This saves one loop over the neighbors of particle r during the calculation of b_r.

Finally, we never want that a pressure increase of particle r leads to a density increase of the same particle for since this would mean that the solver computes attracting forces between rigid particles that have a current density that is below their rest density. To prevent this, we clamp the computed b_r to be at least 0.

C Rendering

We would like to acknowledge the software we use to render the scenes. The surface mesh generation is done using PreonLab by FIFTY2 Technology GmbH [2019]. PreonLab was also used to render the rising sphere and moored buoys scenes. The valley scene was rendered using Houdini by Side Effects Software [2019]. All the other scenes were rendered using the Cycles renderer in Blender by the Blender Online Community [2019].

References

Blender Online Community (2019). Blender. `http://www.blender.org`.

FIFTY2 Technology GmbH (2019). PreonLab. `https://fifty2.eu/`.

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